

ORDER STRUCTURE OF α^* -UNIFORMITIES

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The aim of this paper is to introduce and investigate α^* -uniformity on a non-empty set X using α^* -covers of X in L -fuzzy set theory. Equivalent conditions for α^* -uniformity and basis to α^* -uniformity, respectively, are obtained. The topology generated by α^* -uniformity, called α^* -topology, is also given. Various examples of α^* -uniformity on X are given and the α^* -topology induced by them are also seen. It is shown that the family of all α^* -uniformities on a non-empty set X forms a complete lattice.

1. INTRODUCTION

Various topological structures on a non-empty set, such as topology, uniformity, proximity, merotopy, contiguity and generalizations and variations of these concepts have been created to handle problems of “topological” nature. Generalizations of proximity, merotopy and contiguity in the L -fuzzy theory can be seen in [6, 7, 8, 12, 13].

The notion of uniform spaces was introduced by Weil in 1937 in [15] as a generalization of a metric space. The approach of Weil is called the “entourage” or “surrounding” approach (see [16]). In 1940, Tukey [14] introduced another approach to uniformity through uniform coverings (see also [4]).

The entourage approach and pseudo-metric approach have been generalized to fuzzy situation by Hutton [3], Lowen [10], Höhle [2], Katsaras [5], Liang [9], etc. Covering approach to uniformity in fuzzy theory has been given by Chandrika and Meenakshi [1] using Chang’s definition of fuzzy covers. Covering approach to uniformity on L , where L is a frame (complete lattice satisfying first infinitely distributive law) has also been done (see [11]).

In the present paper, we characterize uniformity in L -fuzzy theory via α^* -covers of a non-empty set X , where L is a completely distributive complete lattice with order reversing involution. In Section 2, some basic definitions and results that are used throughout this paper are collected. In Section 3, the notions of α^* -uniformity, separated α^* -uniformity and basis for α^* -uniformity on X are introduced. An equivalent condition for a family of α^* -covers of X to be a basis for some α^* -uniformity on X is obtained. It is shown that a family \mathcal{U} of α^* -covers

2000 Mathematics Subject Classification: 54E15, 54A40, 54D80.

Key words and phrases: α^* -uniformity, basis, α^* -topology, lattice structure.

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of X is an α^* -uniformity on X if and only if, \mathcal{U} is a filter on X . In Section 4, α^* -topology has been obtained from α^* -uniformity when α is a \bigvee -prime element of L . It has been shown that a separated α^* -uniformity on X induces an α - T_2 L -fuzzy topology on X . Various examples of α^* -uniformity on X and the α^* -topology induced by them are given. It is shown that when α is a \bigvee -prime element of L , α^* -uniformities on X lies between L -fuzzy p.q. metrics and α^* -topologies. For $\alpha = 1$, it is shown that an α^* -topology induced by an α^* -uniformity is different from that of [1]. Lattice structure of a family of α^* -uniformities has been discussed; the supremum and infimum of a family of α^* -uniformities have been given.

2. PRELIMINARIES AND BASIC RESULTS

Let X be a non-empty ordinary set and L be a completely distributive complete lattice with order reversing involution ($' : L \rightarrow L$), largest element 1 and smallest element 0. For definitions of an L -fuzzy set, L -fuzzy point, the relation \leq and induced L -fuzzy mapping $T \rightarrow : L^X \rightarrow L^Y$ see [17]. For $\alpha \in L$, $\mathcal{A} \subset L^X$ is called an α^* -cover of X if and only if, for all $x \in X$, there exists $f \in \mathcal{A}$ such that $\alpha \leq f(x)$. For \mathcal{A}, \mathcal{B} subsets of L^X , we say $\mathcal{A} \wedge \mathcal{B} = \{f \wedge g : f \in \mathcal{A}, g \in \mathcal{B}\}$; \mathcal{A} refines \mathcal{B} ($\mathcal{A} \prec \mathcal{B}$) if and only if, for all $f \in \mathcal{A}$ there exists $g \in \mathcal{B}$ such that $f \leq g$. A mapping $\rho : L^X \times L^X \rightarrow [0, \infty]$ is an L -fuzzy psuedo-quasi (p.q.) metric on X if and only if, it satisfy the following conditions:

- (1) if $f \neq 0$, then $\rho(0, f) = \infty, \rho(f, 0) = \rho(f, f) = 0$;
- (2) $\rho(f, g) \leq \rho(f, h) + \rho(h, g)$;
- (3)(i) if $f \leq g$, then $\rho(f, h) \geq \rho(g, h)$;
- (ii) $\rho(f, \bigvee_{i \in \mathbb{I}} g_i) = \bigvee_{i \in \mathbb{I}} \rho(f, g_i)$, where \mathbb{I} is an arbitrary index set;
- (4) if $\rho(f_i, g) < r \implies g \leq h$, for all $g \in L^X$ and for all $i \in \mathbb{I}$, then the following holds for every $k \in L^X$: $\rho(\bigvee_{i \in \mathbb{I}} f_i, k) < r \implies k \leq h$.

Let ρ be an L -fuzzy p.q. metric on X . Define mapping $N_r : L^X \rightarrow L^X$, for all $r \in \mathbb{R}$ (the set of real numbers), $r > 0$ as follows: for every $f \in L^X$, $N_r(f) = \bigvee \{g \in L^X : \rho(f, g) < r\}$. Call $D_{\rho^*} = \{N_r : r > 0\}$ the associated neighborhood mappings of ρ . For any $t \in L$, the mapping which sends each $x \in X$ to t is denoted by \mathbf{t} . Let $i : L^X \rightarrow L^X$ be a mapping such that $i(\mathbf{1}) = \mathbf{1}; i(f) \leq f; i(f \wedge g) = i(f) \wedge i(g)$. Then i induces an L -fuzzy topology δ on X defined as $\delta = \{f \in L^X : i(f) = f\}$. Further, if $i(i(f)) = i(f)$, then i is an L -fuzzy interior operator on X . An L -filter on X is a non-empty subset \mathcal{F} of L^X satisfying: $\mathbf{0} \notin \mathcal{F}$; if $f \in \mathcal{F}$ and $f \leq g$, then $g \in \mathcal{F}$; if $f \in \mathcal{F}$ and $g \in \mathcal{F}$, then $f \wedge g \in \mathcal{F}$ (see [17]).

A non-zero element $\alpha \in L$ is called \bigvee -prime (\vee -prime) if, for (finite) $\mathcal{M} \subset L$, $\alpha \leq \bigvee \mathcal{M} \implies$ there exists $m \in \mathcal{M}$ such that $\alpha \leq m$. Note that each singleton in $(\mathcal{P}(X), \subset)$ is \bigvee -prime and each atom of a lattice is \bigvee -prime.

3. α^* -UNIFORMITIES

Throughout this paper, we take X a non-empty ordinary set and L a completely distributive complete lattice with order reversing involution ($' : L \rightarrow L$), largest element 1 and smallest element 0; also \mathbb{I} denotes an arbitrary index set and α is a non-zero element of L .

Definition 3.1. Define a relation \leq_α on $L^X \times L^X$ as follows:
 $f \leq_\alpha g$ if and only if, for every $x \in X$, if $x_\alpha \in f$, then $x_\alpha \in g$.

The following observations are obvious:

(i) If $f \leq g$, then $f \leq_\alpha g$.

(ii) The relation \leq_α is reflexive, transitive but not antisymmetric. For example, let $L = X = I \equiv [0, 1]$, $A \equiv [0, \frac{1}{2}]$. Define $f, g \in I^I$ as follows:

for $x \in X$,

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in A, \\ \frac{1}{12}, & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in A, \\ \frac{2}{15}, & \text{otherwise.} \end{cases}$$

Let $\alpha = \frac{1}{3}$. Then $f \leq_\alpha g$ and $g \leq_\alpha f$, but $f \neq g$.

(iii) If $f_i \leq_\alpha g_i$, for every $i \in \mathbb{I}$, then $\bigwedge_{i \in \mathbb{I}} f_i \leq_\alpha \bigwedge_{i \in \mathbb{I}} g_i$. Also if α is \vee -prime, then $\bigvee_{i \in \mathbb{I}} f_i \leq_\alpha \bigvee_{i \in \mathbb{I}} g_i$. Note that if \mathbb{I} is finite and α is \vee -prime, then $\bigvee_{i \in \mathbb{I}} f_i \leq_\alpha \bigvee_{i \in \mathbb{I}} g_i$.

Definition 3.2. Let \mathcal{A}, \mathcal{B} be subsets of L^X . We say that \mathcal{A} α^* -refines \mathcal{B} , denoted by $\mathcal{A} \prec_\alpha \mathcal{B}$, if and only if, for all $f \in \mathcal{A}$, there exists $g \in \mathcal{B}$ such that $f \leq_\alpha g$.

The following observations are obvious:

(i) If $\mathcal{A} \prec \mathcal{B}$, then $\mathcal{A} \prec_\alpha \mathcal{B}$.

(ii) The relation \prec_α is reflexive, transitive but not antisymmetric. For example, $\mathcal{A} \wedge \mathcal{A} \prec_\alpha \mathcal{A} \prec_\alpha \mathcal{A} \wedge \mathcal{A}$ but $\mathcal{A} \neq \mathcal{A} \wedge \mathcal{A}$, in general, if $|\mathcal{A}| \geq 2$.

(iii) If \mathcal{C} is an α^* -cover of X and $\mathcal{C} \prec_\alpha \mathcal{E}$, then \mathcal{E} is also an α^* -cover of X .

Definition 3.3. If \mathcal{C} is an α^* -cover of X , we define

$$st(x_\alpha, \mathcal{C}) = \bigvee \{f \in \mathcal{C} : x_\alpha \in f\},$$

$$st(f, \mathcal{C}) = f \vee (\bigvee \{st(x_\alpha, \mathcal{C}) : x_\alpha \in f\}),$$

$$\mathcal{C}^* = \{st(x_\alpha, \mathcal{C}) : x \in X\}, \text{ and}$$

$$St(\mathcal{C}) = \{st(f, \mathcal{C}) : f \in \mathcal{C}\}$$

Clearly, for any α^* -covers \mathcal{C} and \mathcal{E} of X , $\mathcal{C} \prec_\alpha \mathcal{C}^*$; $\mathcal{C} \prec_\alpha St(\mathcal{C})$; if $\mathcal{C} \prec_\alpha \mathcal{E}$, then $\mathcal{C}^* \prec_\alpha \mathcal{E}^*$, $St(\mathcal{C}) \prec_\alpha St(\mathcal{E})$ and $(\mathcal{C} \wedge \mathcal{E})^* \prec_\alpha \mathcal{C}^* \wedge \mathcal{E}^*$. Also, $st(f, \mathcal{C} \wedge \mathcal{E}) = st(f, \mathcal{C}) \wedge st(f, \mathcal{E})$, for any $f \in L^X$.

Definition 3.4. An α^* -uniformity on X is a non-empty collection \mathcal{U} of α^* -covers of X satisfying the following conditions:

(FU1) if $\mathcal{C} \in \mathcal{U}$ and $\mathcal{E} \in \mathcal{U}$, then $\mathcal{C} \wedge \mathcal{E} \in \mathcal{U}$;

(FU2) if $\mathcal{C} \in \mathcal{U}$ and $\mathcal{C} \prec_\alpha \mathcal{E}$, then $\mathcal{E} \in \mathcal{U}$;

(FU3) for every $\mathcal{C} \in \mathcal{U}$, there exists $\mathcal{E} \in \mathcal{U}$ such that $\mathcal{E}^* \prec_\alpha^* \mathcal{C}$.

We call (X, \mathcal{U}) an α^* -uniform space. Further, \mathcal{U} is a separated α^* -uniformity on X if and only if, it also satisfies the following condition:

(FU4) for every $x, y \in X$, there is an α^* -covering $\mathcal{C} \in \mathcal{U}$ such that every element of \mathcal{C} does not simultaneously contain x_α and y_α .

We call (X, \mathcal{U}) a separated α^* -uniform space. Note that if α is a \vee -prime element of L , then \mathcal{E}^* in (FU3) can be interchanged with $St(\mathcal{E})$.

Definition 3.5. A family \mathcal{B} of α^* -covers of X is a basis for an α^* -uniformity on X if and only if, $\mathcal{B} \subset \mathcal{U}$ and each $\mathcal{C} \in \mathcal{U}$ is α^* -refined by some α^* -cover $\mathcal{E} \in \mathcal{B}$.

Let \mathcal{B} be a family of α^* -covers of X satisfying (FU1) and (FU3). Then is $\mathcal{U} = \{\mathcal{C} : \mathcal{C}_1 \prec_\alpha \mathcal{C}, \text{ for some } \mathcal{C}_1 \in \mathcal{B}\}$ an α^* -uniformity on X .

Definition 3.6. Define a relation \prec_α^* as follows:

$\mathcal{C} \prec_\alpha^* \mathcal{E}$ if and only if, $\mathcal{C}^* \prec_\alpha \mathcal{E}$, for any α^* -covers \mathcal{C} and \mathcal{E} .

Proposition 3.1. Let \mathcal{B} be a family of α^* -covers of X . Then \mathcal{B} is basis for some α^* -uniformity on X if and only if, the following condition is satisfied:

(FB) if $\mathcal{C}_1 \in \mathcal{B}$ and $\mathcal{C}_2 \in \mathcal{B}$, then there exists $\mathcal{C}_3 \in \mathcal{B}$ such that $\mathcal{C}_3 \prec_\alpha^* \mathcal{C}_1 \wedge \mathcal{C}_2$.

Proof. Let \mathcal{B} be basis for some α^* -uniformity \mathcal{U} on X . The proof follows by noting that for α^* -covers $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and \mathcal{C}_4 , if $\mathcal{C}_1 \prec_\alpha \mathcal{C}_2$ and $\mathcal{C}_3 \prec_\alpha \mathcal{C}_4$, then $\mathcal{C}_1 \wedge \mathcal{C}_3 \prec_\alpha \mathcal{C}_2 \wedge \mathcal{C}_4$ and $\mathcal{C}_1^* \prec_\alpha \mathcal{C}_2^*$; also $(\mathcal{C}_3 \wedge \mathcal{C}_4)^* \prec_\alpha \mathcal{C}_3^* \wedge \mathcal{C}_4^*$. Converse follows by noting that if \mathcal{B} satisfies (FB), then $\mathcal{U} = \{\mathcal{C} : \mathcal{C}_1 \prec_\alpha \mathcal{C}, \text{ for some } \mathcal{C}_1 \in \mathcal{B}\}$ is an α^* -uniformity on X .

Lemma 3.1. Let \mathcal{B} be a basis for some α^* -uniformity \mathcal{U} on X . Then $\mathcal{U} = \{\mathcal{C} : \mathcal{C}_1 \prec_\alpha \mathcal{C}, \text{ for some } \mathcal{C}_1 \in \mathcal{B}\} = \{\mathcal{C} : \mathcal{C}_1 \prec_\alpha^* \mathcal{C}, \text{ for some } \mathcal{C}_1 \in \mathcal{B}\}$.

Proof. It is straightforward.

Lemma 3.2. A family \mathcal{U} of α^* -covers of X is an α^* -uniformity on X if and only if, $(\mathcal{U}, \prec_\alpha^*)$ is a filter on X .

Proof. It follows by Lemma 3.1.

Theorem 3.1. *Every separated α^* -uniform structure \mathcal{U} on X induces a separated uniform structure $\mathcal{U}_\alpha^*(\mathcal{U})$ on X , where, $\mathcal{U}_\alpha^*(\mathcal{U}) = \{\mathcal{U}_\alpha^*(\mathcal{A}) : \mathcal{A} \in \mathcal{U}\}$ and $\mathcal{U}_\alpha^*(\mathcal{A}) = \{\{x \in X : f(x) \geq \alpha\} : f \in \mathcal{A}\}$. Conversely, given a separated uniform space (X, μ) , there is a separated α^* -uniform space $(X, \omega_\alpha^*(\mu))$.*

Proof. Let \mathcal{U} be an α^* -uniformity on X . Clearly, $\mathcal{U}_\alpha^*(\mathcal{A})$ is a cover of X , for every $\mathcal{A} \in \mathcal{U}$. To show that $\mathcal{U}_\alpha^*(\mathcal{U})$ is an uniformity on X , it is sufficient to show that $(\mathcal{U}_\alpha^*(\mathcal{U}), \prec^*)$ is a filter on $\mathcal{P}(X)$. Obviously, $\emptyset \notin \mathcal{U}_\alpha^*(\mathcal{U})$ as $\emptyset \notin \mathcal{U}$. Let $\mathcal{U}_\alpha^*(\mathcal{A}), \mathcal{U}_\alpha^*(\mathcal{B}) \in \mathcal{U}_\alpha^*(\mathcal{U})$, where $\mathcal{A}, \mathcal{B} \in \mathcal{U}$. Now, $\mathcal{U}_\alpha^*(\mathcal{A}) \wedge \mathcal{U}_\alpha^*(\mathcal{B}) = \mathcal{U}_\alpha^*(\mathcal{A} \wedge \mathcal{B})$. Since $\mathcal{A} \wedge \mathcal{B} \in \mathcal{U}$, therefore $\mathcal{U}_\alpha^*(\mathcal{A}) \wedge \mathcal{U}_\alpha^*(\mathcal{B}) \in \mathcal{U}_\alpha^*(\mathcal{U})$.

Finally, let $\mathcal{U}_\alpha^*(\mathcal{A}) \prec^* \mathcal{E}$ and $\mathcal{U}_\alpha^*(\mathcal{A}) \in \mathcal{U}_\alpha^*(\mathcal{U})$. That is, $(\mathcal{U}_\alpha^*(\mathcal{A}))^* \prec \mathcal{E}$. Clearly, \mathcal{E} is a cover of X . Let $E \in \mathcal{E}$. Define $f_E : X \rightarrow L$ as follows:

for $x \in X$,

$$f_E(x) = \begin{cases} \alpha, & \text{if } x \in E, \\ 0, & \text{otherwise.} \end{cases}$$

(Note that the map f_E is not uniquely defined). Then $\mathcal{E} = \{\{x \in X : f_E(x) \geq \alpha\} : f_E \in \mathcal{B}\}$, where $\mathcal{B} = \{f_E : E \in \mathcal{E}\}$. Since $\mathcal{U}_\alpha^*(\mathcal{A}) \prec \mathcal{E}$, therefore for all $C \in \mathcal{U}_\alpha^*(\mathcal{A})$, there exists $E \in \mathcal{E}$ such that $C \subset E$. Thus, if $f \in \mathcal{A}$ and $\alpha \leq f(x)$, then $\alpha \leq f_E(x)$, for some $f_E \in \mathcal{B}$ and for all $x \in X$. Consequently, for all $x \in X$, if $x_\alpha \in f$, then $x_\alpha \in f_E$. Therefore, for all $f \in \mathcal{A}$, there exists $f_E \in \mathcal{B}$ such that $f \leq_\alpha f_E$. Thus, $\mathcal{A} \prec_\alpha \mathcal{B}$. Since $\mathcal{A} \in \mathcal{U}$, therefore $\mathcal{B} \in \mathcal{U}$, which gives $\mathcal{E} = \mathcal{U}_\alpha^*(\mathcal{B}) \in \mathcal{U}_\alpha^*(\mathcal{U})$. Further, if \mathcal{U} is a separated α^* -uniformity on X and $x, y \in X$, then there exists $\mathcal{U}_\alpha^*(\mathcal{A}) \in \mathcal{U}_\alpha^*(\mathcal{U})$ such that each element of $\mathcal{U}_\alpha^*(\mathcal{A})$ does not simultaneously contain x and y .

Conversely, let (X, μ) be a uniform space. Define

$$\begin{aligned} \omega_\alpha^*(\mu) &= \{\omega_\alpha^*(\mathcal{A}) : \mathcal{A} \in \mu\}, \\ \omega_\alpha^*(\mathcal{A}) &= \{f_A : \mathcal{A} \in \mathcal{A}\}, \end{aligned}$$

where for $x_i \in X$ ($i \in \mathbb{I}$),

$$f_A(x_i) = \begin{cases} \beta_i \geq \alpha, & \text{if } x_i \in A, \\ \gamma_i \not\geq \alpha, & \text{otherwise,} \end{cases}$$

($\beta_i, \gamma_i \in L$). Since \mathcal{A} is a cover of X , therefore $\omega_\alpha^*(\mathcal{A})$ is an α^* -cover of X , clearly, for all $\mathcal{A} \in \mu$. Then by Lemma 3.2, it is sufficient to show that $(\omega_\alpha^*(\mu), \prec_\alpha^*)$ is a filter on X . Clearly, $\emptyset \notin \omega_\alpha^*(\mu)$. Let $\omega_\alpha^*(\mathcal{A}) \in \omega_\alpha^*(\mu)$ and $\omega_\alpha^*(\mathcal{B}) \in \omega_\alpha^*(\mu)$. Then $\omega_\alpha^*(\mathcal{A}) \wedge \omega_\alpha^*(\mathcal{B}) = \{f_A \wedge f_B : f_A \in \omega_\alpha^*(\mathcal{A}) \text{ and } f_B \in \omega_\alpha^*(\mathcal{B})\} = \{f_{A \cap B} : f_{A \cap B} \in \omega_\alpha^*(\mathcal{A} \wedge \mathcal{B})\}$. Since $\mathcal{A} \wedge \mathcal{B} \in \mu$, therefore $\omega_\alpha^*(\mathcal{A} \wedge \mathcal{B}) \in \omega_\alpha^*(\mu)$.

Finally, let $\omega_\alpha^*(\mathcal{A}) \in \omega_\alpha^*(\mu)$ and $\omega_\alpha^*(\mathcal{A}) \prec_\alpha^* \mathcal{B}$. Then $\omega_\alpha^*(\mathcal{A}) \prec_\alpha \mathcal{B}$. Thus, \mathcal{B} is an α^* -cover of X . Let $\mathcal{C} = \{\{x \in X : g(x) \geq \alpha\} : g \in \mathcal{B}\}$. Since $\omega_\alpha^*(\mathcal{A}) \prec_\alpha \mathcal{B}$, therefore $\mathcal{A} \prec \mathcal{C}$. For $g \in \mathcal{B}$, denote $C_g = \{x \in X : g(x) \geq \alpha\}$. Define $f_{C_g} = g$,

for all $g \in \mathcal{B}$. Thus, $\mathcal{B} = \omega_\alpha^*(\mathcal{C})$. Since $\mathcal{C} \in \mu$, we get $\mathcal{B} \in \omega_\alpha^*(\mu)$. Further, if μ is separated, then $\omega_\alpha^*(\mu)$ is separated, clearly.

4. α^* -UNIFORM TOPOLOGIES AND ORDER STRUCTURE OF α^* -UNIFORMITIES

In this section, we obtain the L -fuzzy topology induced by an α^* -uniformity on X when either α is a \vee -prime element of L or $\alpha = 1$. When $\alpha = 1$, it is shown that the L -fuzzy topology given by an α^* -uniformity on X in our sense is different from that in the sense of [1].

Definition 4.1. *Let α be a \vee -prime element of L and \mathcal{U} be an α^* -uniformity on X . Then $f \in L^X$ is a neighborhood of $x_\alpha \in f$ if and only if, for some α^* -covering $\mathcal{C} \in \mathcal{U}$, $st(x_\alpha, \mathcal{C}) \leq_\alpha f$.*

Theorem 4.1. *Let α be a \vee -prime element of L and \mathcal{U} be an α^* -uniformity on X . Then $\tau = \{f \in L^X : f \text{ is neighborhood of all } x_\alpha \in f\}$ is an L -fuzzy topology on X .*

Proof. Clearly, $\mathbf{0} \in \tau$ and $\mathbf{1} \in \tau$. If $x_\alpha \notin f \wedge g$ for every $x \in X$, then $f \wedge g \in \tau$, vacuously. If $x_\alpha \in f \wedge g$, then $x_\alpha \in f$ and $x_\alpha \in g$. Therefore, there exist $\mathcal{C} \in \mathcal{U}$ and $\mathcal{E} \in \mathcal{U}$ such that $st(x_\alpha, \mathcal{C}) \leq_\alpha f$ and $st(x_\alpha, \mathcal{E}) \leq_\alpha g$. Thus, $st(x_\alpha, \mathcal{C}) \wedge st(x_\alpha, \mathcal{E}) \leq_\alpha f \wedge g$. Consequently, $st(x_\alpha, \mathcal{C} \wedge \mathcal{E}) \leq f \wedge g$. Since $\mathcal{C} \wedge \mathcal{E} \in \mathcal{U}$, therefore $f \wedge g \in \tau$.

Finally, let $\{f_i \in \tau : i \in \mathbb{I}\}$ and $x_\alpha \in \bigvee_{i \in \mathbb{I}} f_i$. Then there exists $i_1 \in \mathbb{I}$ such that $x_\alpha \in f_{i_1}$. Since $f_{i_1} \in \tau$, therefore there exists $\mathcal{C} \in \mathcal{U}$ such that $st(x_\alpha, \mathcal{C}) \leq f_{i_1} \leq \bigvee_{i \in \mathbb{I}} f_i$. Thus, $\bigvee_{i \in \mathbb{I}} f_i \in \tau$.

Definition 4.2. *The above L -fuzzy topology induced by an α^* -uniformity is called α^* -uniform topology on X , or α^* -topology on X simply. Further, an L -fuzzy topological space is called α - T_2 if and only if, for all $x, y \in X$ there exists a neighborhood f of x_α and g of y_α such that $f \wedge g = 0$.*

Observe that, if \mathcal{U} is a separated α^* -uniformity, then τ is an α - T_2 L -fuzzy topological space.

Proposition 4.1. *Let α be a \vee -prime element of L . If τ is an α^* -topology induced by an α^* -uniformity \mathcal{U} on X , then $\tau_\alpha^*(\tau) = \{\{x \in X : f(x) \geq \alpha\} : f \in \tau\}$ is the topology induced by the uniformity $\mathcal{U}_\alpha^*(\mathcal{U})$ on X . Conversely, if τ is induced by an uniformity \mathcal{U} on X , then $\omega_\alpha^*(\tau) = \{\chi_A : A \in \tau\}$ is the α^* -topology induced by the α^* -uniformity $\omega_\alpha^*(\mathcal{U})$ on X .*

Proof. It follows by noting that for any α^* -cover \mathcal{C} of X $st(x, \mathcal{U}_\alpha^*(\mathcal{C})) = \{y \in X : st(x_\alpha, \mathcal{C})(y) \geq \alpha\}$ and for any cover \mathcal{C} of X , $st(x_\alpha, \omega_\alpha^*(\mathcal{C})) \leq_\alpha f_{st(x, \mathcal{C})}$ where the map $f_{st(x, \mathcal{C})}$ is defined in the similar manner as the map f_A in Theorem 3.1.

Proposition 4.2. *Let $\alpha = 1$. Define $i : L^X \rightarrow L^X$ such that $i(f) = \bigvee \{x_1 \in L^X : \text{there exists } \mathcal{C} \in \mathcal{U} \text{ satisfying } st(x_1, \mathcal{C}) \leq_1 f\}$, for every $f \in L^X$. Then i induces an L -fuzzy topology on X .*

Proof. It follows by noting that $i(\mathbf{1}) = \mathbf{1}$; $i(f) \leq f$ and $i(f \wedge g) = i(f) \wedge i(g)$.

Further, note that for any constant function $\mathbf{t} \in L^X$ such that $\mathbf{t} \neq \text{textbf{1}}$, $i(\mathbf{t}) = \mathbf{0}$ in our sense but $i(\mathbf{t}) = \mathbf{t}$ in the sense of [1]. Therefore, our L -fuzzy topology is different from that of [1].

Definition 4.3. *Let \mathcal{U}_X and \mathcal{U}_Y be α^* -uniformities on X and Y , respectively. An L -fuzzy mapping $T^\rightarrow : (X, \mathcal{U}_X) \rightarrow (Y, \mathcal{U}_Y)$ is called α^* -continuous if, f is neighborhood of $x_\alpha \in L^Y \implies T^\leftarrow(f)$ is neighborhood of $T^\leftarrow(x_\alpha) \in L^X$. Further, T^\rightarrow is called α^* -uniformly continuous if, $\mathcal{C} \in \mathcal{U}_Y \implies T^\leftarrow(\mathcal{C}) \in \mathcal{U}_X$.*

Proposition 4.3. *Every α^* -uniformly continuous one-one map $T^\rightarrow : (X, \mathcal{U}_X) \rightarrow (Y, \mathcal{U}_Y)$ is α^* -continuous.*

Proof. Let g be a neighborhood of $T^\rightarrow(x_\alpha) = [T(x)]_\alpha \in L^Y$. Then there exists $\mathcal{C} \in \mathcal{U}_Y$ such that $st(T^\rightarrow(x_\alpha), \mathcal{C}) \leq g$. Since $T^\leftarrow(st(T^\rightarrow(x_\alpha), \mathcal{C})) = st(x_\alpha, T^\leftarrow(\mathcal{C}))$, therefore $st(x_\alpha, T^\leftarrow(\mathcal{C})) \leq T^\leftarrow(g)$. Since T is α^* -uniformly continuous and $\mathcal{C} \in \mathcal{U}_Y$, therefore $T^\leftarrow(\mathcal{C}) \in \mathcal{U}_X$. Thus, $T^\leftarrow(g)$ is a neighborhood of $x_\alpha \in L^X$. Hence, T^\rightarrow is α^* -continuous.

Examples

- (1) The family $\mathcal{D} = \{\mathcal{C} \subset L^X : \mathcal{C} \text{ is an } \alpha^*\text{-cover of } X\}$ is a separated α^* -uniformity called the uniformly discrete α^* -uniformity on X . It follows by noting that for the α^* -cover $\mathcal{E} = \{x_\alpha : x \in X\}$ of X , $\mathcal{E} = \mathcal{E}^*$ and $\mathcal{E} \prec_\alpha \mathcal{C}$, for every α^* -cover \mathcal{C} of X . Further, if α is a \bigvee -prime element of L , then the α^* -uniform topology induced by \mathcal{D} is the discrete L -fuzzy topology on X .
- (2) The family $\mathcal{I} = \{\mathcal{C} \subset L^X : \{\mathbf{1}\} \prec_\alpha \mathcal{C}\}$ is an α^* -uniformity on X , called the indiscrete α^* -uniformity on X . It follows by noting that $st(x_\alpha, \{\mathbf{1}\}) = \mathbf{1}$, for all $x \in X$ and consequently, $\{\mathbf{1}\}$ is the base for the α^* -uniformity \mathcal{I} . Further, if α is a \bigvee -prime element of L , then the α^* -uniform topology induced by \mathcal{I} , called the indiscrete α^* -uniform topology on X , is $\tau(\mathcal{I}) = \{\mathbf{1}\} \cup \{f \in L^X : x_\alpha \notin f, \text{ for all } x \in X\}$.
- (3) Let L be a boolean algebra, α be a \bigvee -prime element of L and \mathcal{F} be an L -filter on X . Then the L -filter α^* -uniformity $\mathcal{U}_{\mathcal{F}}$ on X has all α^* -covers $\{f\} \vee \{x_\alpha : x_\alpha \in f^c\}$ (where $f \in \mathcal{F}$ and f^c is the complement of f in L^X , that is, $f \wedge f^c = 0$ and $f \vee f^c = 1$ [see 17]) as a basis.
- (4) Let α be a \bigvee -prime element of L , then the family $\mathcal{B} = \{\{N_r(x_\alpha) : x \in X\} : r > 0\}$ is a base for an L -fuzzy p.q. metric α^* -uniformity \mathcal{U}_ρ on X , where the α^* -topology τ_ρ on X induced by L -fuzzy p.q. metric is defined as follows: $f \in \tau_\rho$ if and only if, for every $x_\alpha \in f$, there exists

$r > 0$ such that $N_r(x_\alpha) \leq f$. Thus, when α is a \bigvee -prime element of L , an α^* -uniformity lies between an L -fuzzy p.q. metric and an α^* -topology on X .

- (5) Let (X, i) be an L -fuzzy topological space generated by the an L -fuzzy interior operator i on X . Then $\mathcal{U} = \{\mathcal{C} \subset L^X : \mathcal{C} \text{ is an } \alpha^*\text{-cover and } \bigvee_{f \in \mathcal{C}} i(f) \geq x_\alpha, \text{ for every } x \in X\}$ is an α^* -uniformity on X . Further, if α is a \bigvee -prime element of L , then $\mathcal{U} = \{\mathcal{C} \subset L^X : \bigvee_{f \in \mathcal{C}} i(f) \geq x_\alpha, \text{ for every } x \in X\}$ is an α^* -uniformity on X .

Definition 4.4. Let \mathcal{U}_1 and \mathcal{U}_2 be two α^* -uniformities on X . We say that \mathcal{U}_1 is finer than \mathcal{U}_2 , and write $\mathcal{U}_1 < \mathcal{U}_2$, if and only if, $\mathcal{U}_1 \subset \mathcal{U}_2$.

Theorem 4.2. The family of all α^* -uniformities on X forms a complete lattice with respect to the order $<$. The zero of this lattice (i.e. the least element) is the indiscrete α^* -uniformity \mathcal{I} while the unit (i.e. the largest element) is the uniformly discrete α^* -uniformity \mathcal{D} on X . Further, if α is \bigvee -prime element of L , then $\tau(\bigvee_{i \in \mathbb{I}} \mathcal{U}_i) = \bigvee_{i \in \mathbb{I}} \tau(\mathcal{U}_i)$.

Proof. Clearly, the uniformly discrete α^* -uniformity \mathcal{D} on X and the indiscrete α^* -uniformity \mathcal{I} on X are the unit and the zero of the poset $\{\mathcal{U}_i : \mathcal{U}_i \text{ is an } \alpha^*\text{-uniformity on } X, i \in \mathbb{I}\}$. Suppose $\{\mathcal{U}_i : i \in \mathbb{I}\}$ be an arbitrary family of α^* -uniformities on X . Then $\bigcap\{\mathcal{U}_i : i \in \mathbb{I}\}$ is the greatest lower bound of this family, clearly. Let \mathcal{B} consists of all finite meets of elements of $\bigcup\{\mathcal{U}_i : i \in \mathbb{I}\}$ and $\mathcal{C} \in \mathcal{B}$. Without loss of generality, we can assume that $\mathcal{C} = \mathcal{C}_{i_1} \wedge \mathcal{C}_{i_2}$ such that $\mathcal{C}_{i_1} \in \mathcal{U}_{i_1}$ and $\mathcal{C}_{i_2} \in \mathcal{U}_{i_2}$, $i_1, i_2 \in \mathbb{I}$. Therefore, there exists $\mathcal{E}_{i_1} \in \mathcal{U}_{i_1}$ and $\mathcal{E}_{i_2} \in \mathcal{U}_{i_2}$ such that $\mathcal{E}_{i_1}^* \prec_\alpha \mathcal{C}_{i_1}$ and $\mathcal{E}_{i_2}^* \prec_\alpha \mathcal{C}_{i_2}$. Consequently, $\mathcal{E}_{i_1}^* \wedge \mathcal{E}_{i_2}^* \prec_\alpha \mathcal{C}_{i_1} \wedge \mathcal{C}_{i_2} = \mathcal{C}$. Hence, $(\mathcal{E}_{i_1} \wedge \mathcal{E}_{i_2})^* \prec_\alpha \mathcal{C}$. Clearly, $\mathcal{E}_{i_1} \wedge \mathcal{E}_{i_2} \in \mathcal{B}$ being finite meet of elements of $\bigcup_{i \in \mathbb{I}} \mathcal{U}_i$. Thus, \mathcal{B} is a basis for an α^* -uniformity on X . Let \mathcal{U} be the α^* -uniformity generated by the basis \mathcal{B} . Then, by construction, \mathcal{U} is an upper bound of the family $\{\mathcal{U}_i : i \in \mathbb{I}\}$. But any upper bound of this family would have to contain all finite meets of elements of $\bigcup_{i \in \mathbb{I}} \mathcal{U}_i$. So, $\mathcal{U} = \bigvee_{i \in \mathbb{I}} \mathcal{U}_i$.

Conclusion. Notice that we get an α^* -topology on X only when α is a \bigvee -prime element of L and $I \equiv [0, 1]$ does not has any \bigvee -prime element. Thus, the present theory is a unified study of the classical theory, the popular fuzzy theory and the generalized L -fuzzy theory (except the study which involves α^* -topology). Various concepts can be studied using α^* -uniformity on X in L -fuzzy set theory, some of which will appear in our forthcoming papers.

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