COMPLEXIFICATION OF FUZZY SYSTEMS AND THE PROOF OF THE IMPLICATION $0 \Longrightarrow 1$

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Abstract

This work introduces the complexification of fuzzy systems, which seems to be a new approach to the subject matter. This approach then provides us with the powerful mathematical tools of analysis to manipulate fuzzy propositions in an infinite dimensional periodic system. The tools we use enable us to introduce the notions logical independencies, logical bases and logical dimensions that extend the horizon of fuzzy logics.

Complex Fuzzy Sets

It seems reasonable to consider different representations of fuzzy sets. Ofcourse, there are various ways in this approachment (see, for instance, [1], [3], [5], [6], [7] and [8]). This work concerns a representation of fuzzy sets in an infinite dimensional periodic system in order to use tools of analysis besides the algebraic methods.

Throughout the paper $\mathbf{N}, \mathbf{Z}, \mathbf{R}$ and $\mathbf{C}$ will denote the sets of natural numbers, integers, real numbers and complex numbers, in their respective order.

Let X be a universe, whose generic elements are denoted by $x, y,$ etc. In the real case, a fuzzy set is conceived as a function from X into the interval $[0, 1]$. We now first replace the unit interval $[0, 1]$ with the additive group **R mod 1** and then, for further convenience, replace the latter with T, where T is the unit circle $T = \{z : \ z \in \mathbf{C}, \ |z| = 1\}$. We remind that the additive group **R mod 1** is equivalent, as a LCA (locally compact abelian) topological group, to the multiplicative group T. These replacements will enable us to use the algebraic and topological structures of the the unit circle T instead of the sole boolean structure of the unit interval $[0, 1]$. We shall, of course, freely use the terms *fuzzy set* and *fuzzy proposition* interchangeably.

Let F be a set (more precisely a family) of fuzzy sets in X. Then we have

$$\gamma \in F \Rightarrow \gamma : X \to T \tag{1}$$

Fuzzy sets are rich enough to assume further that F separates points on X, i.e., to every pair of distinct points $x_1, x_2 \in X$ there corresponds an element $\gamma \in F$ such that $\gamma(x_1) \neq \gamma(x_2)$.

Since, the set F of fuzzy sets in X is a well-defined crisp set, one may also consider another set (family) $\tilde{F}$ of fuzzy sets in F. As in (1) we write

$$x \in \tilde{F} \Rightarrow x : F \to T \tag{2}$$

Accordingly, for a fixed $\gamma \in F$, while x varies in $\tilde{F}$, the symbol (x, γ) will be interpreted as "γ is a function from $\tilde{F}$ into T." Likewise, for a fixed $x \in \tilde{F}$, while γ varies in F, the symbol (x, γ) will also be interpreted as "x is a function from F into T."

In view of this duality between F and $\tilde{F}$, we may interchangeably use the symbols $x(\gamma) = (x, \gamma) = \gamma(x), \quad \gamma \in F,\ x \in \tilde{F}$, to make it adequate to its context.

We may define the binary operations $\oplus$ and $\tilde{\oplus}$ on F and $\tilde{F}$ by the following relations

$$(\gamma_1 \oplus \gamma_2)(x) = (x, \gamma_1)(x, \gamma_2), \quad \gamma_1, \gamma_2 \in F,\ x \in \tilde{F} \tag{3}$$

and

$$(x_1 \tilde{\oplus} x_2)(\gamma) = (x_1, \gamma)(x_2, \gamma), \quad \gamma \in F,\ x_1, x_2 \in \tilde{F} \tag{4}$$

in their respective order.

We note that, by the duality quoted above, each $x \in X$ can be viewed as a function from F into T. Therefore, it is legitimate to assume that $\tilde{F}$ is chosen to be the set X.

Notice that neither F nor $\tilde{F}$ need to be closed to the operations $\oplus$ and $\tilde{\oplus}$, respectively. To remove these deficits let us now consider the group Γ generated by F and the group G generated by $\tilde{F} = X$ with respect to the binary operations defined in (3) and (4), respectively.

For the sake of simplicity, we shall use the symbol $+$ in place of both operations $\oplus$ and $\tilde{\oplus}$. Thus 0 will stand for the units of the groups G and Γ. Their meanings will always be clear from their context. We also note that 1 will denote the unit of the multiplicative group T. Simple manipulations give the following

$$(\forall \gamma \in \Gamma, \forall x \in G) \Rightarrow (-x, \gamma) = \overline{(x, \gamma)} = (x, -\gamma) \tag{5}$$

We now endow G with the initial (projective) topology τ for the family $\gamma \in \Gamma$. τ is the coarsest topology on G for which all the mappings $\gamma \in \Gamma$ are continuous. Similarly, we endow Γ with the initial (projective) topology η for the family $x \in G$. η is the coarsest topology on Γ for which all the mappings $x \in G$ are continuous.

The following theorems are the cores of our aim and yet their proofs are now easy, and left to the reader.

Theorem 1 . $(G, +, \tau)$ *is a LCA topological group.*

Theorem 2 . $(\Gamma, +, \eta)$ *is a LCA topological group.*

Theorem 3 . *The topological groups* $(\Gamma, +, \eta)$ *and* $(G, +, \tau)$ *are the dual to each other.*

Logical Dependence, Logical Base, Logical Dimension

Let us remind that if one of the dual groups G and Γ is compact then the other is discrete [2]. Assume that G is compact and Γ is discrete. Their Haar measures μ and ν, respectively, can be normalized so that $\mu(G) = 1$ and $\nu(\{\gamma\}) = 1$ for each $\gamma \in \Gamma$. We shall freely use some notions and notations from harmonic analysis. The inner product $< p, q >$ of complex fuzzy sets (propositions) p and q is defined by the equation

$$< p, q > = \int_G p(x)\overline{q(x)}d\mu(x) \tag{6}$$

Definition 1 . *A family V of complex fuzzy sets (complex fuzzy propositions) in Γ is said to be orthonormal provided that the relation*

$$< \gamma_1, \gamma_2 >= \int_G (x, \gamma_1)\overline{(x, \gamma_2)} d\mu(x) = \begin{cases} 1 & if \quad \gamma_1 = \gamma_2 \\ 0 & if \quad \gamma_1 \neq \gamma_2 \end{cases} \tag{7}$$

holds for each pair $\gamma_1, \gamma_2 \in V$.

The class of all orthonormal complex fuzzy propositions is clearly partially ordered by the inclusion relation $\subset$.

Definition 2 . *A complete orthonormal system χ of complex fuzzy propositions is defined to be one that is maximal, in other words, χ is complete if it is not properly contained in any other orthonormal set of complex fuzzy propositions.*

Since Γ is discrete, a complete orthonormal system of complex fuzzy propositions is at most countably infinite.

Definition 3 . *Let S be a finite family of complex fuzzy propositions in Γ. Any sum of the form*

$$f(x) = \sum c_\gamma(-x, \gamma), \ (c_\gamma \in \mathbf{C}, \ \gamma \in S)$$

is said to be a finite combination of complex fuzzy propositions of S.

Definition 4 . *Let A be a family of complex fuzzy propositions in Γ, not necesserily finite. Then the set $\Gamma(A)$ of all combinations of all finite family of A is called the logical span of A.*

Note that $\Gamma(A)$ is the minimal linear space which contains A. We now extend the notion of complex proposition as follows.

Definition 5 . *Let A be any subset of Γ. Any element $f \in \Gamma(A)$ is said to be a complex fuzzy proposition.*

If V is orthonormal, we associate with each complex fuzzy proposition f a function $\hat{f}$ on the set V, defined by

$$\hat{f}(\gamma) =< f, \gamma > \tag{8}$$

The following two lemmas follow from the definition.

Lemma 1 . *We have*

$$A \subset B \Rightarrow \Gamma(A) \subset \Gamma(B)$$

for all $A, B \subset \Gamma$.

Lemma 2 . *The relation*

$$\Gamma(A \cap B) = \Gamma(A) \cap \Gamma(B)$$

holds for all $A, B \subset \Gamma$.

Definition 6 . *The logical base of a complex fuzzy proposition f, denoted by B_f, is defined to be the set*

$$B_f = \cap\{A \mid f \in \Gamma(A)\} \tag{9}$$

Note that, the base B_f of a complex fuzzy proposition f is the subset of Γ with the property that

$$\widehat{f}(\gamma) = \begin{cases} < f, \gamma > \neq 0 & if \quad \gamma \in B_f \\ 0 & if \quad \gamma \notin B_f \end{cases} \tag{10}$$

Let 0 be the falsity, i.e., the zero proposition on G defined by $0(x) = (x, 0) = 0$ for all $x \in G$. It is obvious that the logical base of 0 is empty, i.e., $B_0 = \phi$ (empty set) and that the logical base of a complex fuzzy proposition f is the minimal set A in Γ for which $f \in \Gamma(A)$.

Definition 7 . *Two complex fuzzy propositions f and g are logically independent iff $B_f \cap B_g = \phi$.*

Definition 8 . *The logical dimension of a complex fuzzy proposion f is defined to be the cardinal number of its logical base B_f.*

Clearly, the cardinality $card(B_f)$ of B_f satisfies the relation $0 \leq cardB_f \leq +\infty$ for each complex fuzzy proposition f, i.e., the logical base B_f of f is at most countably infinite. If $card(B_f) = n$ for some $n \in \mathbf{N}$ then we say that the logical dimension of f is finite.

Lemma 3 . *$B_\gamma = \{\gamma\}$ for all $\gamma \in \Gamma$, where $\{\gamma\}$ stands for the set whose sole element is γ.*

It follows that $B_\gamma \cap B_\eta = \phi$ for all $\gamma, \eta \in \Gamma$ with $\gamma \neq \eta$, meaning that different complex fuzzy propositions in Γ are logically independent.

An equivalence relation $\equiv$ on $\omega(G)$ of all complex fuzzy propositions on G can be defined by

$$g \equiv f \quad \Leftrightarrow \quad B_g = B_f. \tag{11}$$

We let $[f]$ denote the equivalence class of f in $\omega(G)$, and $\Omega(G)$ stand for the set of equivalence classes $[f]$, $f \in \omega(G)$.

It is easy to see from (10) that $h \in [f]$ iff there exists a number $a_\gamma \neq 0$ such that $\widehat{f}(\gamma) = a_\gamma \widehat{h}(\gamma)$ for each $\gamma \in \Gamma$.

Definition 9 . *We define the binary operation $\wedge$ on $\Omega(G)$ by the relation*

$$[f] \wedge [g] = [f] \cap [g]$$

$[f] \wedge [g]$ is the equivalence class whose base is $B_f \cap B_g$. In particular, we define the operation $f \wedge g$ to be any element in $[f] \wedge [g]$.

The Implication

Definition 10 . *Let f and g be two complex fuzzy propositions. We say that f implies g, written $f \Rightarrow g$, iff $B_f \subset B_g$.*

Lemma 4 . *We have the implication $0 \Rightarrow f$ for any complex fuzzy proposition f.*

Proof. If $f = 0$, there is nothing to prove. So, let us assume that $f \neq 0$ in which case the logical base B_f of f cannot be empty. But, then the relation $B_0 = \phi \subset B_f$ complex fuzzy proposition follows proving the required result.

This is a simple proof of the important assumption in boolean logic.

Remark. If the base B_f of a complex fuzzy proposition f is equal to an orthonormal system χ in Γ, i.e., its logical base is maximal, then it has the complete expansion of the form

$$f(x) = \sum (f, \gamma)(-x, \gamma) \; , x \in G \; , \; \gamma \in \chi \tag{12}$$

meaning that f has the maximal truth value. Otherwise, the expansion (12) is incomplete. Hence its truth value is not maximal and proportinal to its logical base.

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