Nonadditive Measures and an Extension Theorem

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The aim of the present paper is to establish interrelationship among continuity conditions for a nonnegative extended real-valued function μ defined on an effect algebra. Examples and counterexamples are given to illustrate various situations arising in this study.

1. Introduction

In 1992, Kôpka defined the D-posets of fuzzy sets in [17], which is closed under the formations of differences of fuzzy sets, while studying the axiomatical systems of fuzzy sets. The structure of a D-poset supports a noncommutative measure theory and allows the solution of some problems of noncommutative probability theory, including some problems of theory of quantum measurement. In the "quantum probability theory" one assume the occurrence of noncompatible events, that is, events that can be tested separately but not simultaneously. Thus the set of noncompatible events does not fulfil the axioms of Boolean algebra. Therefore Boolean algebra is replaced by orthomodular lattice or poset [13]. The concept of an effect algebra (which is a common generalization of orthomodular lattices and MV-algebras) has been introduced by Foulis and Bennet [4] as an algebraic structure providing an instrument for studying quantum effects that may be unsharp. For a list of nice examples of effect algebras we refer to [6] and for some of its properties we refer also to [4] and [5]. Effect algebras, which are essentially equivalent to D-posets, were introduced as the carriers of states or probability measure in the Quantum Physics [7], in Mathematical Economics ([10], [11]) and in Fuzzy Theory ([8], [12], [14-16], [19-21]). The categorical equivalence of D-posets and effect algebras is discussed in [9]. Nonadditive measures appear today in many branches of pure mathematics with many important applications ([18], see also [22]).

Avallone [3] gave the concept of upper continuous (also called continuous from below) and lower continuous (also called continuous from above) functions defined on a D-lattice in the context of subadditive measures. In the present paper,

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notions of weakly null-additive function, null-additive function and different types of continuities of a function defined on an effect algebra E are introduced and interrelationship among these conditions are studied. Prerequisites and some basic results on effect algebras are collected in Section 2, which have been extensively used in the subsequent sections. In Section 3, we introduced nonadditive measure μ defined on an effect algebra E, with values in $[0, \infty]$. It is shown that being a weakly null-additive function, is a weaker condition than null-additivity. We have proved for a function μ defined on a σ-complete effect algebra E that (i) if μ (with $\mu(0) = 0$) is strongly order continuous, then μ is order continuous (ii) if μ is monotone and order continuous, then μ is exhaustive (a characterization of exhaustivity in terms of a μ-Cauchy sequences is established and used to prove converse of this statement); converse of both of these statements need not be true, which has been established through counterexamples. We have proved that every null-additive and order continuous function is null-continuous and also every weakly null-additive, strongly order continuous function is null-continuous; two different techniques are used to obtain a suitable decreasing sequence from a given increasing sequence while proving these results. Here also converse of both of these statements need not be true, moreover converses in each case is established by adding suitable necessary conditions.

2. Preliminaries and Basic Results

Throughout the paper, $E = (E; \oplus, 0, 1)$ denotes, in general, an *Effect algebra* (see [1-6, 9]). In every effect algebra E, a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a. We say that two elements $a, b \in E$ are *orthogonal* and we write $a \perp b$, if $a \oplus b$ exists. If $a \oplus b = 1$, then b is *orthocomplement* of a and write $b = a^{\perp}$. It is clear that $1^{\perp} = 0$, $(a^{\perp})^{\perp} = a$, $a \perp 0$ and $a \oplus 0 = a$, for all $a \in E$. Also for $a, b \in E$, define $a \leq b$ if there exists $c \in E$ such that $a \perp c$ and $a \oplus c = b$. It may be proved that $\leq$ is a partial ordering on E and $0 \leq a \leq 1$; $a \leq b \Leftrightarrow b^{\perp} \leq a^{\perp}$ and $a \leq b^{\perp} \Leftrightarrow a \perp b$ for $a, b \in E$. If $a \leq b$, the element $c \in E$ such that $c \perp a$ and $a \oplus c = b$ is unique, and satisfies the condition $c = (a \oplus b^{\perp})^{\perp}$. In this case we write $c = b \ominus a$. If $(E, \leqslant)$ is a lattice, we say that the effect algebra E is a *lattice effect algebra*, or a *D-lattice*. For elements a, b of a D-lattice, we set $a \Delta b = (a \vee b) \ominus (a \wedge b)$ [3].

For $a_1, \ldots, a_n \in E$, we inductively define $a_1 \oplus \ldots \oplus a_n = (a_1 \oplus \ldots \oplus a_{n-1}) \oplus a_n$, provided that the right hand side exists. The definition is independent on permutation of the elements. A finite subset $\{a_1, \ldots, a_n\}$ of E is said to be *orthogonal* if $a_1 \oplus \ldots \oplus a_n$ exists. A sequence $\{a_n\}$ in E is called orthogonal if, for every n, $\bigoplus_{i \leqslant n} a_i$ exists. If, moreover $\sup_n \bigoplus_{i \leqslant n} a_i$ exists, the *sum* $\bigoplus_{n \in \mathbb{N}} a_n$ of an orthogonal sequence $\{a_n\}$ in E is defined as $\sup_n \bigoplus_{i \leqslant n} a_i$; $\mathbb{N}$ denotes the set of all natural numbers. An effect algebra E is called a *σ-complete effect algebra* if every orthogonal sequence in E has its sum [1, 2, 9].

Let us recall the following results which we shall use in the sequel.

2.1 [2]. Let E be a lattice effect algebra (or D-lattice). We write $a_n \uparrow a$ (respectively, $a_n \downarrow a$) whenever $\{a_n\}$ is an increasing sequence in E and $a = \sup_n a_n$ (respectively, $\{a_n\}$ is an decreasing sequence in E and $a = \inf_n a_n$).

2.2 [1]. Let $a, b, c \in E$ such that $a \perp b$ and $b \leq c$. Then $a \oplus b \leq c$ if and only if $a \leq c \ominus b$.

2.3 [1]. (i) Let $\{a_1, ..., a_n\} \subseteq E$ be orthogonal. If $1 \leq k \leq n$, then $\{a_1, ..., a_k\}$ and $\{a_{k+1}, ..., a_n\}$ are orthogonal and $\bigoplus_{i=1}^n a_i = \bigoplus_{i=1}^k a_i \oplus \bigoplus_{i=k+1}^n a_i$.

(ii) Let $\{a_n\}$ be an orthogonal sequence in E and $A, B \subseteq \mathbb{N}$ disjoint such that $a = \bigoplus_{n\in A} a_n$ and $b = \bigoplus_{n\in B} a_n$ exist. Then $a \perp b$ and $a \oplus b = \bigoplus_{n\in A\cup B} a_n$.

(iii) Let $\{a_n\}$ be an orthogonal sequence in E and $A, B \subseteq \mathbb{N}$ be such that $B \subseteq A$ and there exist $a = \bigoplus_{n\in A} a_n$ and $b = \bigoplus_{n\in B} a_n$. Then $\bigoplus_{n\in A\setminus B} a_n$ exists and we have $\bigoplus_{n\in A\setminus B} a_n = a \ominus b$.

2.4 [2]. (i) Let $\{a_0, a_1, \ldots a_n\}$ be in E with $a_0 \leq a_1 \leq \ldots \leq a_n$ and let $b_i = a_i \ominus a_{i-1}$ for every $i \in \{1, 2, \ldots, n\}$. Then $\{b_1, b_2, \ldots b_n\}$ is orthogonal and $b_1 \oplus b_2 \oplus \ldots \oplus b_n = a_n \ominus a_0$.

(ii) Let E be a σ-complete effect algebra. If $\{a_n\}$ is an increasing (respectively, decreasing) sequence, then $\sup_n a_n$ (respectively, $\inf_n a_n$) exists.

2.5. ([2], [9]) Assume that a, b, c are elements of an effect algebra E.

(i) If $a \leqslant b$, then $b = a \oplus (b \ominus a)$.

(ii) If $a \leqslant b$, then $b \ominus a \leqslant b$ and $b \ominus (b \ominus a) = a$.

(iii) If $a \perp b$, then $a \leq a \oplus b$ and $(a \oplus b) \ominus a = b$.

(iv) If $b \leqslant c$ and $a \leq c \ominus b$, then $b \leq c \ominus a$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.

(v) If $a \leqslant c$ and $b \leqslant c$, then $c \ominus a = c \ominus b$ if and only if $a = b$.

(vi) If $a \leqslant b \leqslant c$, then $(c \ominus b) \oplus a$ exists and $(c \ominus b) \oplus a = c \ominus (b \ominus a)$.

(vii)If $a \perp b$ and $a \oplus b \leq c$, then $c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$.

(viii) If $a \leqslant b \leqslant c$, then $(c \ominus b) \leqslant (c \ominus a)$ and $(c \ominus a) \ominus (c \ominus b) = (b \ominus a)$.

(x) If $a \leqslant b \leqslant c$, then $(b \ominus a) \leqslant (c \ominus a)$ and $(c \ominus a) \ominus (b \ominus a) = (c \ominus b)$.

2.6. ([3], [9]) Assume that a, b, b_n $(n \in \mathbb{N})$ are elements of a D-lattice E.

(i) If $b_n \downarrow b$ and $a \perp b_n$ for each n, then $a \oplus b_n \downarrow a \oplus b$.

(ii) If $b_n \downarrow b$ and $a \geq b_n$ for each n, then $a \ominus b_n \uparrow a \ominus b$.

(iii) If $b_n \downarrow b$ and $a \leq b_n$ for each n, then $b_n \ominus a \downarrow b \ominus a$.

(iv) If $b_n \uparrow b$ and $a \geq b_n$ for each n, then $a \ominus b_n \downarrow a \ominus b$.

3. Interrelationship Among Continuity Conditions

Let E be an effect algebra and μ be a $[0, \infty]$-valued function defined on E.

Definition 3.1. *μ is called a nonadditive function if it satisfies the following conditions:*

(i) $\mu(0) = 0$; (ii) (monotone) if $a, b \in E$, $a \leq b$, then $\mu(a) \leq \mu(b)$.

Definition 3.2. *A function μ is called weakly null-additive, if $\mu(b \oplus c) = 0$ provided $b, c \in E$, $b \perp c$ and $\mu(b) = \mu(c) = 0$.*

Definition 3.3. *A function μ is called null-additive, if $\mu(b \oplus c) = \mu(b)$ provided $b, c \in E$, $b \perp c$ and $\mu(c) = 0$.*

Observe that, μ is null-additive if and only if $\mu(b \ominus c) = \mu(b)$ provided $b, c \in E$, $c \leq b$ and $\mu(c) = 0$. Also, if a function μ is null-additive, then it is weakly null-additive, but the following example shows that converse need not be true.

Example 3.1. *Let $E_1 = \{0, a, b, c, d, e, 1\}$. Let us define: $a \oplus b = b \oplus a = c$, $b \oplus c = c \oplus b = a \oplus d = d \oplus a = e \oplus e = 1$ and let $x \oplus 0 = 0 \oplus x$ for all $x \in E_1$. Then E_1 is an effect algebra. Define a function μ_1 on E_1 as follows: $\mu_1(x) = 1$ if $x \in \{a, e\}$, and $\mu_1(x) = 0$ if $x \in \{0, b, c, d, 1\}$. Then μ_1 is weakly null-additive but not null-additive.*

Example 3.2. *Let $E_2 = \{0, a, b, c, 1\}$. Let us define: $a \oplus b = b \oplus a = c$, $b \oplus c = c \oplus b = a \oplus a = 1$ and let $x \oplus 0 = 0 \oplus x$ for all $x \in E_2$. Then E_2 is an effect algebra. Define a function μ_2 on E_2 as follows: $\mu_2(x) = 1$ if $x \in \{c, 1\}$, and $\mu_2(x) = 0$ if $x \in \{0, a, b\}$. Then μ_2 is not weakly null-additive (and so μ_2 is not null-additive). Further, define a function μ_3 on E_3 by, $\mu_3(x) = 1$ if $x \in \{a, c, 1\}$, and $\mu_3(x) = 0$ if $x \in \{0, b\}$. Then μ_3 is null-additive.*

Definition 3.4. *For a function μ, we say that μ is*

(i) continuous from below, if $a_n \uparrow a$, $a_n \in E$ $(n \in \mathbb{N})$, $a \in E$, then $\lim_{n\to\infty} \mu(a_n) = \mu(a)$,

(ii) continuous from above, if $a_n \downarrow a$, $a_n \in E$ $(n \in \mathbb{N})$, $a \in E$, then $\lim_{n\to\infty} \mu(a_n) = \mu(a)$,

The function μ is called continuous if it is both continuous from below and continuous from above,

(iii) order continuous, if $a_n \downarrow 0$, $a_n \in E$ $(n \in \mathbb{N})$, then $\lim_{n\to\infty} \mu(a_n) = 0$,

(iv) strongly order continuous, if $\{a_n\}$ is a decreasing sequence in E and $\mu(\bigwedge_{n=1}^{\infty} a_n) = 0$ (provided $\bigwedge_{n=1}^{\infty} a_n$ exists), then $\lim_{n\to\infty} \mu(a_n) = 0$,

(v) null-continuous, if $\{a_n\}$ is an increasing sequence in E and $\mu(a_n) = 0$ $(n \in \mathbb{N})$, then $\mu(\bigvee_{n=1}^{\infty} a_n) = 0$, (provided $\bigvee_{n=1}^{\infty} a_n$ exists),

(vi) exhaustive, if $\{a_n\}$ is an orthogonal sequence in E, then $\lim_{n\to\infty} \mu(a_n) = 0$,

(vii) A function μ is called autocontinuous from below, if $\lim_{n\to\infty} \mu(c_n) = 0$ implies $\lim_{n\to\infty} \mu(b \ominus c_n) = \mu(b)$ whenever $b \in E$, $\{c_n\}$ is a sequence in E, and $c_n \leq b$ $(n \in \mathbb{N})$,

(vii) A function μ is called autocontinuous from above, if $\lim_{n\to\infty} \mu(c_n) = 0$ implies $\lim_{n\to\infty} \mu(b \oplus c_n) = \mu(b)$ whenever $b \in E$, $\{c_n\}$ is a sequence in E, $b \perp c_n$ $(n \in \mathbb{N})$,

The function μ is called autocontinuous if it is both autocontinuous from below and autocontinuous from above.

Observe that if μ is autocontinuous from below or autocontinuous from above, then it is null-additive.

Example 3.3. *Let* $E_3 = \{0, 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{p}, ...\}$. *Let us define: for each* $\frac{1}{p}$, $0 \oplus \frac{1}{p} = \frac{1}{p}$, $\frac{1}{p} \oplus \frac{1}{p} = 1$, $0 \oplus 1 = 1$ *and if* $p \neq q$, $\frac{1}{p} \oplus \frac{1}{q}$ *is undefined. Then* E_3 *is a* σ*-complete effect algebra. Define functions* μ_4, μ_5 *and* μ_6 *on* E_3 *as follows:*

(I) $\mu_4(1) = 1$, *and* $\mu_4(x) = 0$ *if* $x \in \{0, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{p}, ...\}$.

(II) $\mu_5(x) = 0$, *for all* $x \in E_3$.

(III) $\mu_6(0) = 0$, *and* $\mu_6(x) = 1$ *if* $x \in \{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{p}, ...\}$.

Then we have,

(i) μ_4, μ_5 and μ_6 are continuous from below;

(ii) μ_4 and μ_5 are continuous from above, but μ_6 is not continuous from above;

(iii) μ_4 and μ_5 are order continuous, but μ_6 is not order continuous;

(iv) μ_4 and μ_5 are strongly order continuous, but μ_6 is not strongly order continuous;

(v) μ_4, μ_5 and μ_6 are null-continuous;

(vi) μ_4 and μ_5 are exhaustive, but μ_6 is not exhaustive;

(vii) μ_5 are μ_6 are autocontinuous, but μ_4 is not autocontinuous;

Proposition 3.1. *Let* μ *be a null-additive and continuous from above function defined on a* σ*-complete D-lattice* E. *If* $a \in E$, *then* $\lim_{n\to\infty} \mu(a \oplus b_n) = \mu(a)$ *for any decreasing sequence* $\{b_n\}$ *in* E *for which* $\lim_{n\to\infty} \mu(b_n) = 0$ *and* $a \perp b_n$ $(n \in \mathbb{N})$.

Proof. Let $\{b_n\}$ be a decreasing sequence in E such that $\lim_{n\to\infty} \mu(b_n) = 0$ and let $a \in E$ with $a \perp b_n$ $(n \in \mathbb{N})$. By 2.4(ii), put $b = \bigwedge_{n=1}^{\infty} b_n$. Then $\mu(b) = \lim_{n\to\infty} \mu(b_n) = 0$. Now, since $a \perp b_n$ $(n \in \mathbb{N})$ and $b_n \downarrow b$, so by 2.6(i), we have $a \oplus b_n \downarrow a \oplus b$. Hence by the null-additivity of μ, we get $\lim_{n\to\infty} \mu(a \oplus b_n) = \mu(a \oplus b) = \mu(a)$.

Proposition 3.2. *Let* μ *be a null-additive and continuous function defined on a* σ*-complete D-lattice* E. *If* $a \in E$, *then* $\lim_{n\to\infty} \mu(a \ominus b_n) = \mu(a)$ *for any decreasing sequence* $\{b_n\}$ *in* E *for which* $\lim_{n\to\infty} \mu(b_n) = 0$ *and* $b_n \leq a$ $(n \in \mathbb{N})$.

Proof. Let $\{b_n\}$ be a decreasing sequence in E such that $\lim_{n\to\infty} \mu(b_n) = 0$ and let $a \in E$ with $b_n \leq a$, $(n \in \mathbb{N})$. Putting $b = \bigwedge_{n=1}^{\infty} b_n$, we obtain that $b \in E$ and $\mu(b) = \lim_{n\to\infty} \mu(b_n) = 0$. Again, since $b_n \downarrow b$ and $b_n \leq a$, so by 2.6(ii), we get $a \ominus b_n \uparrow a \ominus b$. Hence by the null-additivity of μ, we obtain $\lim_{n\to\infty} \mu(a \ominus b_n) = \mu(a \ominus b) = \mu(a)$.

Proposition 3.3. *Let* μ *be an order continuous and autocontinuous from above* (*respectively, autocontinuous from below*) *function on a D-lattice* E. *Then* μ *is continuous from above* (*respectively, continuous from below*).

Proof. Let $a_n \downarrow a$, $a_n \in E$ $(n \in \mathbb{N})$, $a \in E$. By 2.6(iii), we have $a_n \ominus a \downarrow 0$. Since μ is order continuous, therefore $\lim_{n\to\infty} \mu(a_n \ominus a) = 0$. Now, since $a \leq a_n$, so by 2.5(i) we obtain $a_n = a \oplus (a_n \ominus a)$. Hence $\lim_{n\to\infty} \mu(a_n) = \lim_{n\to\infty} \mu(a \oplus (a_n \ominus a)) = \mu(a)$, as μ is autocontinuous from above, and the result follows.

Remark 3.1. *Converse of the above Proposition 3.3 need not be true, as μ_4 is continuous from above but not autocontinuous.*

Proposition 3.4. *Let μ be function defined on a σ-complete effect algebra E.*
(i) If μ is continuous from above, then it is strongly order continuous.
(ii) If μ is strongly order continuous, with $\mu(0) = 0$, then it is order continuous.
(iii) If μ is continuous from below, then it is null-continuous.

Proposition 3.5. *Let μ be monotone and order continuous function defined on a σ-complete effect algebra E. Then μ is exhaustive.*

Proof. Let $\{a_n\}$ be an orthogonal sequence in E. In view of 2.3(ii), we have $\bigoplus_{i=n}^{\infty} a_i \downarrow 0$ as $n \to \infty$. By the order continuity of μ, we get $\lim_{n\to\infty} \mu(\bigoplus_{i=n}^{\infty} a_i) = 0$. Since $a_n \leq \bigoplus_{i=n}^{\infty} a_i$, so we obtain $\lim_{n\to\infty} \mu(a_n) = 0$.

Remark 3.2. *Converse of Proposition 3.5 need not be true: consider the effect algebra E_1 of Example 3.1. Define a function μ_7 on E_1 as follows: $\mu_7(x) = 1$ if $x \in \{a, b, 1\}$, and $\mu_7(x) = 0$ if $x \in \{0, c, d, e\}$. Then μ_7 is exhaustive but not monotone.*

Theorem 3.1. *Let μ be a nonadditive function defined on a σ-complete effect algebra E. Then μ is exhaustive if and only if $\mu(a_n \Delta a_m) \to 0$ as $n, m \to \infty$ for any monotone sequence $\{a_n\}$ in E (which is called μ-Cauchy).*

Proof. Let μ be exhaustive. Suppose that $\{a_n\}$ is an increasing sequence but not μ-Cauchy. Then there exists $\varepsilon > 0$ and a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}$ such that $\mu(a_{n_{k+1}} \ominus a_{n_k}) = \mu(a_{n_{k+1}} \Delta\, a_{n_k}) \geq \varepsilon$, $\forall\ n \geq k$. Put $b_k = a_{n_{k+1}} \ominus a_{n_k} (k \in \mathbb{N})$. Then by 2.4(i), $\{b_k\}$ is an orthogonal sequence and $\lim_{k\to\infty} \mu(b_k) \geq \varepsilon$, which contradicts that μ is exhaustive. Conversely, if $\{a_n\}$ is an orthogonal sequence in E, then $\bigoplus_{n\in\mathbb{N}} a_n$ exists and by 2.3(ii), we have $b_n = \bigoplus_{i=n}^{\infty} a_n \downarrow 0$ as $n \to \infty$. Thus $\{b_n\}$ is μ-Cauchy and so we obtain, $\lim_{n\to\infty} \mu(a_n) = \lim_{n\to\infty} \mu(b_n \ominus b_{n+1}) = \lim_{k\to\infty} \mu(b_n \Delta\, b_{n+1}) = 0$.

Proposition 3.6. *Let μ be a nonadditive, exhaustive and continuous from below function defined on a σ-complete D-lattice E. Then μ is order continuous.*

Proof. Let $a_n \downarrow 0$, $a_n \in E$ $(n \in \mathbb{N})$. For any fixed $n \in \mathbb{N}$, using 2.6(ii) we get $a_n \ominus a_m \uparrow a_n$ as $m \to \infty$ (as $a_n \geq a_m$ for $m \geq n$). Consequently, $\lim_{m\to\infty} \mu(a_n \ominus a_m) = \mu(a_n)$. Now, for a given $\varepsilon > 0$, choose $n_1, n_2 \in \mathbb{N}$ such that $\mu(a_n) < \mu(a_n \ominus a_m) + \frac{\varepsilon}{2}$, for all $m \geq n_1$ and $\mu(a_n \Delta a_m) < \frac{\varepsilon}{2}$, for all $n, m \geq n_2$. Now, for $m \geq n \geq n_0$ $(n_0 = \max\{n_1, n_2\})$ we obtain, $\mu(a_n) < \mu(a_n \ominus a_m) + \frac{\varepsilon}{2} = \mu(a_n \Delta a_m) + \frac{\varepsilon}{2} < \varepsilon$, showing that $\lim_{n\to\infty} \mu(a_n) = 0$.

Aliter. Let us suppose on the contrary. Then there exists an $\varepsilon_0 > 0$ and a sequence $\{a_n\}$ with $a_n \downarrow 0$, and $\mu(a_n) > \varepsilon_0$ $(n \in \mathbb{N})$. Since (by using 2.6(ii)), $(a_k \ominus a_n) \uparrow a_k$ for $n \geq k$ as $n \to \infty$, for any fixed k and μ is continuous from below, we get $\lim_{n\to\infty}\mu(a_1 \ominus a_n) = \mu(a_1) > \varepsilon_0$. Thus there exists an $n(1)$ such that $\mu(a_1 \ominus a_{n(1)}) > \varepsilon_0$. In the same way, since $\lim_{n\to\infty}\mu(a_{n(1)} \ominus a_n) = \mu(a_{n(1)}) > \varepsilon_0$, choose $n(2) > n(1)$ with $\mu(a_{n(1)} \ominus a_{n(2)}) > \varepsilon_0$. Thus, we obtain a subsequence $\{n(k)\}$ of $\mathbb{N}$ such that $n(1) < n(2) < \ldots$ and $\mu(a_{n(k)} \ominus a_{n(k+1)}) > \varepsilon_0$. Put $b_k = a^{\perp}_{n(k)} \ominus a^{\perp}_{n(k+1)}$ $(k \in \mathbb{N})$, using 2.4(i) and 2.5(viii), we obtain an orthogonal sequence $\{b_k\}$ of elements in L, with $\mu(b_k) > \varepsilon_0$, which contradicts the fact that μ is exhaustive.

Remark 3.3. *Exhaustivity of μ is an essential condition for the above Proposition 3.6. The function μ_6 is monotone and continuous from below, but not order continuous* (*observe that μ_6 is not exhaustive*).

Proposition 3.7. *Let μ be a null-additive and order continuous function defined on a σ-complete D-lattice E. Then μ is strongly order continuous.*

Proof. Let $\{a_n\}$ be a decreasing sequence in E and $\mu(\bigwedge_{n=1}^{\infty} a_n) = 0$, ($\bigwedge_{n=1}^{\infty} a_n$ exists by 2.4(ii)). Put $a = \bigwedge_{n=1}^{\infty} a_n$. Since $a_n \downarrow a$, then by 2.6(iii) we have $a_n \ominus a \downarrow 0$. Now, since $a \leq a_n$ $(n \in \mathbb{N})$, so $a_n = a \oplus (a_n \ominus a)$ and by the null-additivity and order continuity of μ we obtain

$$\lim_{n\to\infty} \mu(a_n) = \lim_{n\to\infty} \mu(a \oplus (a_n \ominus a)) = \lim_{n\to\infty} \mu(a_n \ominus a) = 0.$$

Remark 3.4. *Converse of the above Proposition 3.7 need not be true, as μ_4 is strongly order continuous but not null-additive.*

Remark 3.5. *In view of Proposition 3.6, 3.7 we obtain that a function μ which is null-additive, exhaustive and continuous from below defined on a σ-complete D-lattice E, then μ is strongly order continuous; however, converse of this statement need not be true, as μ_4 is strongly order continuous but not null-additive.*

Theorem 3.2. *Let μ be a null-additive and order continuous function defined on a σ-complete effect algebra E. Then μ is null-continuous.*

Proof. Let $\{a_n\}$ be an increasing sequence in E and $\mu(a_n) = 0$ $(n \in \mathbb{N})$. Since $\{a_n\}$ is an increasing sequence take, $b_n = a_n \ominus a_{n-1}$ with $a_0 = 0$. By 2.4(i), we obtain $\{b_n\}$ is orthogonal and $\bigoplus_{i\leq n} b_i = a_n$. Since E is σ-complete, we get $a = \sup_n(\bigoplus_{i\leq n} b_i) = \sup_n a_n$ exists, and so $c_n = a \ominus (\bigoplus_{i\leq n} b_i)$ exists. Now, by 2.5(viii), $\{c_n\}$ is a decreasing sequence. Let $d \leq c_n$ $(n \in \mathbb{N})$. Then, by 2.5(iv), $\bigoplus_{i\leq n} b_i \leq a \ominus d$, for each n. Taking the supremum over n, we obtain $a \leq a \ominus d$. Now, by 2.5(v), we get $d = 0$, so $c_n \downarrow 0$. Since $\bigoplus_{i\leq n} b_i \leq a$, so $a = \bigoplus_{i\leq n} b_i \oplus (a \ominus (\bigoplus_{i\leq n} b_i)) = (\bigoplus_{i\leq n} b_i) \oplus c_n$. Now, by the order continuity and null-additivity of μ, we obtain $\mu(a) = \lim_{n\to\infty} \mu((\bigoplus_{i\leq n} b_i) \oplus c_n) = \lim_{n\to\infty} \mu(c_n) = 0$. Thus $\mu(\bigvee_{n=1}^{\infty} a_n) = 0$, hence the result follows.

Remark 3.6. *Converse of Theorem 3.2 need not be true, as μ_4 is null-continuous but not null-additive.*

Theorem 3.3. *Let μ be a weakly null-additive and strongly order continuous function defined on a σ-complete D-lattice E. Then μ is null-continuous.*

Proof. Let $\{a_n\}$ be an increasing sequence in E and $\mu(a_n) = 0$ $(n \in \mathbb{N})$. Since E is σ-complete, by 2.3, we obtain $a = \sup_n a_n$, and hence $a_n \uparrow a$. Define a subsequence $\{a_{n_m}\}$ of $\{a_n\}$ as follows: Let $n_1 = 1$. For $m \in \mathbb{N}$, we have $\mu(a_{n_m}) = 0$. By 2.6(iv) we obtain $a \ominus a_n \downarrow 0$ and since $a_{n_m} \leq a_n \leq a$, for all $n \geq m$, so by 2.5(vi) we have $(a \ominus a_n) \oplus a_{n_m}$ exists. Now, since $(a \ominus a_n) \oplus a_{n_m} \downarrow a_{n_m}$ as $n \to \infty$, (by use of 2.6(i)) therefore from strong order continuity, we can choose $n_2 > n_1 = 1$ such that $\mu(a_1 \oplus (a \ominus a_{n_2})) < 1$ for all $n \geq 1$. Similarly, $n_3 > n_2 > n_1 = 1$ such that $\mu(a_{n_2} \oplus (a \ominus a_{n_3})) < \frac{1}{2}$ for all $n \geq 1$. Thus, we can choose n_{m+1} such that $n_{m+1} > n_m$ and $\mu(a_{n_m} \oplus (a \ominus a_{n_{m+1}})) < \frac{1}{m}$. Since $\{a_{n_m}\}$ is an increasing sequence, put $b_{n_i} = a_{n_i} \ominus a_{n_{i-1}}$ with $a_{n_0} = 0$. By 2.4(i), we have $\{b_{n_i}\}$ is orthogonal and $\bigoplus_{k \leq n_i} b_k = a_{n_i}$. Since E is σ-complete, we have $a = \sup_{n_i} \bigoplus_{k \leq n_i} b_k = \sup_{n_i} a_{n_i}$. Thus by 2.3(ii), we define

$$c = \bigoplus_{i=1}^{\infty}(a_{n_{2i}} \ominus a_{n_{2i-1}}) \text{ and } d = a_{n_1} \oplus (\bigoplus_{i=1}^{\infty}(a_{n_{2i+1}} \ominus a_{n_{2i}})).$$

Since $c \perp d$ and $d \leq a$, so by 2.2, we have $c \oplus d \leq a$ if and only if $c \leq a \ominus d$. By 2.5(ii) and (vii), we have

$$c \leq (a \ominus (a_{n_3} \ominus a_{n_2})) \ominus ((a_{n_1} \oplus (a_{n_5} \ominus a_{n_4}) \oplus \ldots) \leq a \ominus (a_{n_3} \ominus a_{n_2}) = a_{n_2} \oplus (a \ominus a_{n_3}).$$

Similarly, we obtain $c \leq a_{n_4} \oplus (a \ominus a_{n_5})$. Hence, for every i, we have $c \leq (a_{n_{2i}} \oplus (a \ominus a_{n_{2i+1}}))$. Using similar arguments, we obtain, for every i, $d \leq (a_{n_{2i-1}} \oplus (a \ominus a_{n_{2i}}))$. Since μ is monotone, we get for every i,

$$\mu(c) \leq \mu(a_{n_{2i}} \oplus (a \ominus a_{n_{2i+1}})) < \frac{1}{2i} \text{ and } \mu(d) \leq \mu(a_{n_{2i-1}} \oplus (a \ominus a_{n_{2i}})) < \frac{1}{2i-1}.$$

Hence, we have $\mu(c) = 0$ and $\mu(d) = 0$. Now, since μ is weakly null-additive, so $\mu(a) = \mu(c \oplus d) = 0$. Therefore, μ is null-continuous.

Remark 3.7. *Converse of Theorem 3.3 need not be true, as μ_4 is null-continuous but not weakly null-additive.*

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