# Lattice of fuzzy congruences on completely 0-simple semigroups *†‡

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**Abstract** The lattice of fuzzy congruences on a completely 0-simple semigroup is developed in the present paper. We prove that the lattice of fuzzy congruences on a completely 0-simple semigroup is semimodular.

**Keywords** Completely 0-simple semigroup; Fuzzy equivalence relation; Fuzzy congruence; Fuzzy linked triple

**AMS Classfication** **20M10, 20M99**

## 1 Introduction and prelimilaries

Crisp congruence relations on semigroups play an important role in studying algebraic structures of semigroups [2-4]. Fuzzy relations on a set have also been studied since the concept of fuzzy relations was introduced by Nemitz [8]. Fuzzy equivalent relations and fuzzy ordering have been studied in the literature of fuzzy mathematics [6, 8]. Samhan among others proved that normal fuzzy subgroups and fuzzy congruence relations determined each other in groups[2]. Moreover, proved that the lattice of all fuzzy congruences on a group $G$ (resp. ring $R$) is ismorphic to the lattice of fuzzy normal subgroups of $G$ (resp. fuzzy ideals of $R$) [2,4,9]. For a semigroup $S$, Samhan obtained that the lattice of fuzzy congruences on a semigroup $S$ is a complete lattice. In particularly, if $S$ is a group, then the lattice of all fuzzy congruences on $S$ is a modular lattice [2]. For some special algebraic structures of semigroups, such as inverse semigroups, Kuroki characterized fuzzy congruences on an inverse semigroup by the concept of fuzzy congruence pair. For some special fuzzy congruences on a semigroup, for example, fuzzy Rees congruence, Xie characterized $RC$-semigroups by fuzzy

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Rees congruences. Recently, Wang[5] introduced the concept of fuzzy linked triple of a fuzzy congruence on a completely 0-simple semigroup and proved that there is a one-one correspondence between fuzzy proper congruences on a completely 0-simple semigroup and fuzzy linked triples. As continuation of the study undertaken by Wang[5], the lattice of fuzzy congruences on a completely 0-simple semigroup is developed in the present paper. We prove that the lattice of fuzzy congruences on a completely 0-simple semigroup is semimodular.

All fuzzy relations on a set $X$ are maps: $X \times X \longmapsto I$. If $\theta$ and $\phi$ are two fuzzy relations on a set $X$, then $\theta \leq \phi$ means that $\theta(x,y) \leq \phi(x,y)(\forall x, y \in X)$. Their composition is denoted by $\theta \circ \phi$ and defined as

$$\theta \circ \phi(x,y) = \bigvee_{z \in X} (\theta(x,z) \wedge \phi(z,y)) \qquad (\forall x, y \in X)[2].$$

A fuzzy relation $\theta$ on $X$ is called a fuzzy equivalent relation on $X$ if (i) $\theta(x,x) = 1(\forall x \in X)$ (reflexive); (ii) $\theta(x,y) = \theta(y,x)(\forall x, y \in X)$ (symmetric); (iii) $\theta \circ \theta \leq \theta$ (transitive). A fuzzy equivalent relation $\theta$ on a semigroup $S$ is called fuzzy compatible if $(\forall x, y, z, t \in S) \quad \theta(x,y) \wedge \theta(z,t) \leq \theta(xz, yt)$. $\theta$ is called fuzzy left ( right) compatible if $(\forall x, y, z \in S) \qquad \theta(x,y) \leq \theta(zx, zy)$, $(\theta(x,y) \leq \theta(xz, yz).)$ A fuzzy equivalent relation $\theta$ on a semigroup $S$ is called fuzzy congruence if it is fuzzy compatible.

We denote the set of all fuzzy congruences on a semigroup $S$ by $FC(S)$. Then Samhan [2] proved that $FC(S)$ is a complete lattice with respect to "$\cdot$" and "$+$" defined as follows:

$$(\forall \theta, \phi \in FC(S)) \qquad \theta + \phi = (\theta \vee \phi)^e, \theta \cdot \phi = \theta \wedge \phi.$$

Furthermore, $\theta + \phi = (\theta \circ \phi)^\infty$. Let elements $\nabla_S$ and $\triangle_S$ be fuzzy congrunces on $S$ defined by $\nabla_S(x,y) = 1$ for all $x, y \in S$, and $\triangle_S(x,y) = 0$ if $x \neq y$ and $\triangle_S(x,y) = 1$ if $x = y$. Then it is clear that $\nabla_S$ and $\triangle_S$ are the greatest element and least element of the lattice $(FC(S), +, \cdot)$ respectively.

For other definition and terminology not given in this paper, we employ those in [1,2].

## 2 Lattice of fuzzy congruences on completely 0-simple semigroups

In this section, $M(x,y)$ will denote the minimum of $x$ and $y$, $N$ denote the set of all positive integers. From now on, S means a completely 0-simple semigroup $\mathcal{M}^0[G; I, \Lambda; P]$. If $p_{\lambda i}, p_{\mu i}, p_{\lambda j}$ and $p_{\mu j}$ are all non-zero, then we write $q_{\lambda\mu ij} = p_{\lambda i} p_{\mu i}^{-1} p_{\mu j} p_{\lambda j}^{-1}$. Moreover, we can show easily the following:

$$q_{\lambda\mu ij} q_{\lambda\mu jk} = q_{\lambda\mu ik}, \quad p_{\lambda i}^{-1} q_{\lambda\mu ij}^{-1} p_{\lambda i} p_{\mu i}^{-1} q_{\mu\nu ij}^{-1} p_{\mu i} = p_{\lambda i}^{-1} q_{\lambda\nu ij}^{-1} p_{\lambda i} \qquad (2.1).$$

An equivalence relation $\varepsilon_I$ on I is defined by the rule that

$$(i,j) \in \varepsilon_I \ \ iff \ \ \{\lambda \in \Lambda : p_{\lambda i} = 0\} = \{\lambda \in \Lambda : p_{\lambda j} = 0\},$$

and an equivalence relation $\varepsilon_\Lambda$ on $\Lambda$ is defined by the rule that

$$(\lambda,\mu) \in \varepsilon_\Lambda \ \ iff \ \ \{i \in I : p_{\lambda i} = 0\} = \{i \in I : p_{\mu i} = 0\}.$$

Let $\rho$ be a fuzzy congruence on S. $\rho$ is called *proper* if it satisfies the following statements:$\rho(0,0) = 1$, $\rho(0,(i,a,\lambda)) = \rho((i,a,\lambda),0) = 0$, and $\rho((i,a,\lambda),(j,b,\mu)) = 0$ whenever $(i,j) \notin \varepsilon_I$ or $(\lambda,\mu) \notin \varepsilon_\Lambda$. By $\varepsilon_I$ and $\varepsilon_\lambda$, we define fuzzy relations $\rho_I$ on $I$ and $\rho_\lambda$ on $\lambda$ as follows:

$$\rho_I((i,j)) = \begin{cases} \bigwedge_{p_{\lambda i}\neq 0} \rho((i,p_{\lambda i}^{-1},\lambda),(j,p_{\lambda j}^{-1},\lambda)) & , \ \text{if} \ \ (i,j) \in \varepsilon_I \\ 0 & , \ \text{if} \ \ (i,j) \notin \varepsilon_I \end{cases}$$

$$\rho_\Lambda((\lambda,\mu)) = \begin{cases} \bigwedge_{p_{\lambda i}\neq 0} \rho((i,p_{\lambda i}^{-1},\lambda),(i,p_{\mu i}^{-1},\mu)) & , \ \text{if} \ \ (\lambda,\mu) \in \varepsilon_\Lambda \\ 0 & , \ \text{if} \ \ (\lambda,\mu) \notin \varepsilon_\Lambda \end{cases}$$

It is easily to see that $\rho_I$ and $\rho_\Lambda$ are two fuzzy equivalence relations on $I$ and $\Lambda$, respectively. Let $\rho$ be a fuzzy proper congruence on S. We define a fuzzy subset $N_\rho$ of G by $N_\rho : G \longrightarrow [0,1]$, $a \mapsto \rho((1,a,1),(1,e,1))$, where $e$ is the identity of G. Then $N_\rho$ is a fuzzy normal subgroup of $G$ [5].

**Definition 2.1** [5] A fuzzy triple $(N,\mathcal{S},\mathcal{T})$ consisting of a fuzzy normal subgroup $N$ of $G$, a fuzzy equivalence relation $\mathcal{S}$ on $I$ and a fuzzy equivalence relation $\mathcal{T}$ on $\Lambda$ will be called linked if

(i) $\mathcal{S} \subseteq \varepsilon_I, \mathcal{T} \subseteq \varepsilon_\Lambda$;

(ii) If $i,j$ in I and $\lambda,\mu$ in $\Lambda$ are such that $p_{\lambda i}, p_{\mu i}, p_{\lambda j}$ and $p_{\mu j}$ are all non-zero, then $\mathcal{N}(q_{\lambda i\mu j}) > \mathcal{S}(i,j) \vee \mathcal{T}(\lambda,\mu)$.

**Theorem 2.2** [5] *Let $S = \mathcal{M}^0[G;I,\Lambda;P]$ be a completely 0-simple semigroup. Then the mapping $\Gamma : \rho \mapsto (N_\rho,\rho_I,\rho_\Lambda)$ is an order-preserving bijection from the set of fuzzy proper congruences on $S$ onto the set of fuzzy linked triples. Conversely, If $(N,\mathcal{S},\mathcal{T})$ is a fuzzy linked triple of $S = \mathcal{M}^0[G;I,\Lambda;P]$ we shall write the fuzzy congruence $(N,\mathcal{S},\mathcal{T})\Gamma^{-1}$ corresponding to the fuzzy triple as $[N,\mathcal{S},\mathcal{T}]$ (with square brackets). Thus we have that $\rho = [N_\rho,\rho_I,\rho_\Lambda]$.*

**Lemma 2.3** [1] *The lattice $(\varepsilon(X),\subseteq,\cap,\vee)$ of fuzzy equivalences on a set $X$ is semimodular.*

**Lemma 2.4** [1] *The direct product $L_1\times L_2\times\cdots\times L_n$ of semimodular lattices $L_1,L_2,\cdots,L_n$ is semimodular.*

**Lemma 2.5** [1] *Every modular lattice is semimodular.*

Now we shall discuss the lattice of fuzzy congruences on a completely 0-simple semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$. First, we prove a useful lemma as follows:

**Lemma 2.6** *If $\rho$ and $\delta$ are fuzzy proper congruences on a completely 0-simple semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$, then*

$$\rho \cap \delta = [N_\rho \cap N_\delta, \rho_I \cap \delta_I, \rho_\Lambda \cap \delta_\Lambda], \quad \rho \vee \delta = [N_\rho \circ N_\delta, \rho_I \vee \delta_I, \rho_\Lambda \vee \delta_\Lambda].$$

**Proof.** To prove the first of these statements, notice first that $(N_\rho \cap N_\delta, \rho_I \cap \delta_I, \rho_\Lambda \cap \delta_\Lambda)$ is a fuzzy linked triple. For it is clear that $\rho_I \cap \delta_I \subseteq \varepsilon_I$ and $\rho_\Lambda \cap \delta_\Lambda \subseteq \varepsilon_\Lambda$, and if $i, j$ in I and $\lambda, \mu$ in $\Lambda$ are such that $p_{\lambda i}, p_{\mu i}, p_{\lambda j}$ and $p_{\mu j}$ are all non-zero, then

$$\begin{aligned} (N_\rho \cap N_\delta)(q_{\lambda\mu ij}) &= N_\rho(q_{\lambda\mu ij}) \wedge N_\delta(q_{\lambda\mu ij}) \\ &\geq \rho_I(i,j) \wedge \delta_I(i,j) = (\rho_I \cap \delta_I)(i,j), \end{aligned}$$

and similarly, $(N_\rho \cap N_\delta)(q_{\lambda\mu ij}) \geq (\rho_\Lambda \cap \delta_\Lambda)(\lambda, \mu)$. Hence we have

$$(N_\rho \cap N_\delta)(q_{\lambda\mu ij}) \geq (\rho_I \cap \delta_I)(i,j) \vee (\rho_\Lambda \cap \delta_\Lambda)(\lambda, \mu)$$

Thus there is a fuzzy congruence $[N_\rho \cap N_\delta, \rho_I \cap \delta_I, \rho_\Lambda \cap \delta_\Lambda]$ which, by Theorem 2.2, is contained in $\rho$ and in $\delta$ and is the largest congruence with these properties.

To prove the second part, we first show that $(N_\rho \circ N_\delta, \rho_I \vee \delta_I, \rho_\Lambda \vee \delta_\Lambda)$ is a fuzzy linked triple. Since $\rho_I, \delta_I \subseteq \varepsilon_I$ we have that $\rho_I \vee \delta_I \subseteq \varepsilon_I$, and similarly $\rho_\Lambda \vee \delta_\Lambda \subseteq \varepsilon_\Lambda$. It is not difficult to show that $N_\rho \circ N_\delta$ is the smallest fuzzy normal subgroup of G containing both $N_\rho$ and $N_\delta$. If $i, j$ in I and $\lambda, \mu$ in $\Lambda$ are such that $p_{\lambda i}, p_{\mu i}, p_{\lambda j}$ and $p_{\mu j}$ are all non-zero, then we have

$$\begin{aligned} &(\rho_I \vee \delta_I)(i,j) \\ =\ &sup\{(\rho_I \circ \delta_I)^n(i,j); n \in N\} \\ =\ &sup\{sup\{M((\rho_I \circ \delta_I)(i,i_1), \cdots, (\rho_I \circ \delta_I)(i_{n-1},j)); i_1, \cdots, i_{n-1} \in I\}; n \in N\} \\ \leq\ &sup\{sup\{M((N_\rho \circ N_\delta)(q_{\lambda\mu i i_1}), \cdots, (N_\rho \circ N_\delta)(q_{\lambda\mu i_{n-1} i_1})); i_1, \cdots, i_{n-1} \in I\}; n \in N\} \\ =\ &sup\{sup\{(N_\rho \circ N_\delta)(q_{\lambda\mu ij}); i_1, \cdots, i_{n-1} \in I\}; n \in N\} (\textit{by the equation } (2.1)) \\ =\ &(N_\rho \circ N_\delta)(q_{\lambda\mu ij}). \end{aligned}$$

Also,

$$\begin{aligned} &(\rho_\Lambda \vee \delta_\Lambda)(\lambda, \mu) \\ =\ &sup\{(\rho_\Lambda \circ \delta_\Lambda)^n(\lambda, \mu); n \in N\} \end{aligned}$$

$$= sup\{sup\{M((\rho_\Lambda \circ \delta_\Lambda)(\lambda,\lambda_1),\cdots,(\rho_\Lambda \circ \delta_\Lambda)(\lambda_{n-1},\mu));\lambda_1,\cdots,\lambda_{n-1}\in\Lambda\};n\in N\}$$
$$= sup\{sup\{M((N_\rho \circ N_\delta)(q_{\lambda\lambda_1 ij}),\cdots,(N_\rho \circ N_\delta)(q_{\lambda_{n-1}\mu ij}));\lambda_1,\cdots,\lambda_{n-1}\in\Lambda\};n\in N\}$$
$$= sup\{sup\{M((N_\rho \circ N_\delta)(p_{\lambda i}^{-1}q_{\lambda\lambda_1 ij}^{-1}p_{\lambda i}),\cdots,(N_\rho \circ N_\delta)(p_{\lambda_{n-1}i}^{-1}q_{\lambda_{n-1}\mu ij}^{-1}p_{\lambda_{n-1}i}));$$
$$\lambda_1,\cdots,\lambda_{n-1}\in\Lambda\};n\in N\}$$
$$= sup\{sup\{(N_\rho \circ N_\delta)(p_{\lambda i}^{-1}q_{\lambda\mu ij}^{-1}p_{\lambda i});\lambda_1,\cdots,\lambda_{n-1}\in\Lambda\};n\in N\}$$
(*by the equation* (2.1))
$$= (N_\rho \circ N_\delta)(q_{\lambda\mu ij}).$$

From the above statements, we have $(N_\rho \circ N_\delta)(q_{\lambda\mu ij}) \geq (\rho_I \vee \delta_I)(i,j) \vee (\rho_\Lambda \vee \delta_\Lambda)(\lambda,\mu)$.

We have shown that $(N_\rho \circ N_\delta, \rho_I \vee \delta_I, \rho_\Lambda \vee \delta_\Lambda)$ is a fuzzy linked triple, and it thus follows from Theorem 2.2 that $[N_\rho \circ N_\delta, \rho_I \vee \delta_I, \rho_\Lambda \vee \delta_\Lambda] = \rho \vee \delta$.

We can now establish the most wonderful result in this paper:

**Theorem 2.7** The lattice of fuzzy congruences on a completely 0-simple semigroup is semimodular.

**Proof.** Let $S = \mathcal{M}^0[G;I,\Lambda;P]$. Let $(\mathcal{N},\cap,\circ)$ be the lattice of all fuzzy normal subgroups of G, let $(\mathcal{P},\cap,\vee)$ be the lattice consisting of all fuzzy equivalences on $I$ contained in $\varepsilon_I$ and let $(\mathcal{Q},\cap,\vee)$ be the lattice consisting of all fuzzy equivalences on $\Lambda$ contained in $\varepsilon_\Lambda$. Let $\mathcal{X} = \mathcal{N} \times \mathcal{P} \times \mathcal{Q}$ be the direct product of these three lattices. By Lemma 2.6 the subset $\mathcal{Y}$ of $\mathcal{X}$ consisting of all fuzzy linked triples $(N,\mathcal{S},\mathcal{T})$ in $\mathcal{X}$ is a sublattice of $\mathcal{X}$. The effect of Theorem 2.2 is to give us an isomorphim $\Gamma$ between the lattice $(\mathcal{K},\cap,\vee)$ of fuzzy proper congruences on $S$ and the lattice $\mathcal{Y}$. Now $\mathcal{X}$ is semimodular by Lemmas 2.3,2.4,2.5. If $(N_1,\mathcal{S}_1,\mathcal{T}_1)$ and $(N_2,\mathcal{S}_2,\mathcal{T}_2)$ belong to $\mathcal{Y}$. Similar to Lemma 3.6.3 in [1], we can prove that $(N_1,\mathcal{S}_1,\mathcal{T}_1)$ covers $(N_2,\mathcal{S}_2,\mathcal{T}_2)$ in $\mathcal{Y}$ if and only if $(N_1,\mathcal{S}_1,\mathcal{T}_1)$ covers $(N_2,\mathcal{S}_2,\mathcal{T}_2)$ in $\mathcal{X}$. It will follow that $\mathcal{Y}$ is semimodular. Thus the lattice $\mathcal{K}$ of fuzzy proper congruences on $S$ is semimodular, since it is isomorphic to $\mathcal{Y}$. The lattice $C(S)$ of all fuzzy congruences on $S$ is obtained from $\mathcal{K}$ by adjoining a single extra maximum element, the fuzzy universal congruence $\nabla_S$ of $S$. Since the maximum element can never figure either as $\rho$ or as $\delta$ in the hypothesis A if $\rho \succ \rho\cap\delta$ and $\delta \succ \rho\cap\delta$ A appearing in the definition of semimodularity, and so the adjunction of this element does not destroy the semimodularity property of the lattice. The proof of Theorem 2.7 is complete.

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