On the representation of Łukasiewicz-Moisil algebra by L-fuzzy sets

Lemnaouar Zedam and Abdelaziz Amroune *

Abstract

The aim of the paper is the elaboration of a representation theory of θ-valued involutive Łukasiewicz-Moisil algebra, the concept of L-fuzzy sets playing the role that the notion of field of sets plays for the representation of Boolean algebras. This theory provides both a semantic interpretation of Łukasiewicz many valued logic and logical basis to L-fuzzy set theory.

1 Introduction.

In 1941 [10], Moisil introduced the notion of Łukasiewicz-Moisil algebra under the name Łukasiewician algebras, as an algebraic counterpart of the Łukasiewicz many-valued logics. The theory of Łukasiewicz-Moisil algebras has been developed both as a tool for studying certain non-classical logics and as an algebraic theory having its own interest; besides, it is now considered one of the fundamental formalizations of fuzzy logic. The reader is referred e.g. to [1, 2, 3, 4, 5, 13, 15, 16].

Not long after Zadeh published his important paper " fuzzy sets " [18] Goguen published the paper " $L-$fuzzy sets " [8] such that the concept of $L-$fuzzy sets (briefly LFSs) is a generalization of the concept of fuzzy sets and takes the latter as a special case when $L = [0, 1]$. There are several different kinds of understanding and employment of the concept of an $L-$fuzzy set distinguished by how to specify the lattice L (see eg., [7, 17, 19]). An $L-$fuzzy set on a universe X (LFS, for short) is a function $A : X \longrightarrow L$. One of the understandings of L is that $(L, \leq_L, *)$ is a complete lattice equipped with a multiplication operator $*$ satisfying certain conditions as shown in the initial paper of Goguen [8]. On the other hand, many investigation require L to be more general and posses no extra structure other than lattice structure, and the second understanding of the meaning of L is that $(L, \leq_L, \aleph)$ is a complete lattice with an order-reversing involution $\aleph$.

*Supported by Laboratory of Pure and Applied Mathematics, M'sila University.
2000 *Mathematics Subject Classification.* 03G10, 03G20, 06B23, 03E72.
Key words and phrases. Łukasiewicz-Moisil algebra, L-fuzzy set, Representation theorem.

The second understanding of L is widely used by researchers in the field of fuzzy algebra. For example, if we consider the meaning of L is that L is a chain (ie., totally ordered set) with largest element $\top$ and least element $\perp$, Moisil [14] showed that the set $F_L(X)$ of $L-$fuzzy sets on a universe X can be equipped with a structure of $\bar{\theta}-$valued Łukasiewicz-Moisil algebra, where $\bar{\theta}$ is the order type of L. Conversely, Ponasse [15] proved that every θ-valued Łukasiewicz-Moisil algebra $(\mathcal{L}, J, (\varphi_\alpha)_{\alpha\in J^0})$ where θ is the order type of the chain J can be embedded in an algebra of the form $F_L(X)$, where L is the chain of all ideals of $J-\{0\}$; thus L is of type $\bar{\theta}\geq\theta$. Rudeanu [16] proved that under certain hypotheses on $C(\mathcal{L})$ the Boolean sublattice of complemented elements of $\mathcal{L}$, every θ-valued Łukasiewicz-Moisil algebra $(\mathcal{L}, J, (\varphi_\alpha)_{\alpha\in J^0})$ can be embedded in an algebras of the form $F_L(X)$, where L is the initial chain J; thus L is of type $\bar{\theta}=\theta$. In [15] Ponasse has also studied the representability of involutive θ-valued Łukasiewicz-Moisil algebra $(\mathcal{L}, J, (\varphi_\alpha)_{\alpha\in J^0}, (\Psi_\alpha)_{\alpha\in J^1}, N, n)$ as algebras of the form $F_L(X)$, where L is a regular chain equipped with an order-reversing involution $\aleph$. This problem was also investigated in [4, 5], where Coulon proved that under certain conditions on J and $C(\mathcal{L})$, every involutive θ-valued Łukasiewicz-Moisil algebra can be embedded in an algebras of the form $F_L(X)$, where L is the MacNeille completion of J; thus L is of type $\bar{\theta}\geq\theta$.

In the present paper we want to study, in a new way, the representability of involutive θ-valued Łukasiewicz-Moisil algebra as algebras of the form $F_L(X)$, where L has an order type $\bar{\theta}=\theta$ (i.e., there exists an isomorphism between L and J).

In the next section, we will remind the reader what an involutive $\theta-$valued Łukasiewicz-Moisil algebra is [11 - 13] and we recall definitions and results on $L-$fuzzy sets [8] that are needed. Section 3 describes our main results.

2 Preliminaries

2.1 Łukasiewicz-Moisil algebra

In this section, we give some notations, definitions and results on which our work in this paper is based. We note by:

$$J^0 = J-\{0\}\,, J^1 = J-\{1\}\,, J^{01} = J-\{0,1\}\,.$$

Definition 1 *(Moisil [11 - 13]) An involutive $\theta-$valued Łukasiewicz-Moisil algebra is a 6-tuple $(\mathcal{L}, J, (\varphi_\alpha)_{\alpha\in J^0}, (\Psi_\alpha)_{\alpha\in J^1}, n, N)$, where:*
1) $(\mathcal{L},\vee,\wedge,0,1)$ is a distributive lattice with largest element 1 and least element 0. The Boolean sublattice of complemented elements of $\mathcal{L}$ is denoted $C(\mathcal{L})$;
2) J is a chain with largest element 1 and least element 0 and whose order type is θ ;

3) n is an order reversing involution in J;
4) N is an order reversing involution in £ satisfying for all $x \in C(£)$, $N(x) = \bar{x}$, ($\bar{x}$ is the complement element of x) ;
5) $(\varphi_\alpha)_{\alpha \in J^0}$ is a family of morphisms £ $\longrightarrow C(£)$ such that: $\forall \alpha, \beta \in J^0$:
  (i) $\varphi_\alpha(0) = 0, \quad \varphi_\alpha(1) = 1$
  (ii) If $\alpha \leq \beta$ then $\varphi_\beta \leq \varphi_\alpha$
  (iii) $\varphi_\alpha \circ \varphi_\beta = \varphi_\beta$
  (iv) If $\varphi_\alpha(x) = \varphi_\alpha(y)$ for all $\alpha \in J^0$ then $x = y$ (Moisil's determination principle);
6) $(\Psi_\alpha)_{\alpha \in J^1}$ is a family of morphisms £ $\longrightarrow C(£)$ such that:
  (i) $\Psi_\alpha \leq \varphi_\alpha, \quad$ for all $\alpha \in J^{01}$
  (ii) $\varphi_\alpha N = N\Psi_{n\alpha}, \quad$ for all $\alpha \in J^0$

We use the same symbols $\leq, 0, 1$ for the partial order, least and greatest elements of J, respectively, and for those of £. We shall freely note $x \leq y$ or $y \geq x$ in the sequel.

Example 1. Let $(B, \wedge, \vee, \bar{\ }, 0, 1)$ be a boolean algebra. The set

$$D(B) = \{f/f : J \longrightarrow B, i \leq j \Rightarrow f(j) \leq f(i)\}$$

of all decreasing functions from J to B can be made into an involutive θ-valued Łukasiewicz-Moisil algebra where the operations of the lattice $(D(B), \wedge, \vee, 0, 1)$ are defined for all $i \in J$ by
$(f \wedge g)(i) = f(i) \wedge g(i)$
$(f \vee g)(i) = f(i) \vee g(i)$
$0(i) = 0$, $1(i) = 1$.
And
$(\varphi_\alpha f)(i) = f(n\alpha), \qquad$ for all $\alpha \in J^0$
$(\Psi_\alpha f)(i) = \overline{f(n\alpha)}, \qquad$ for all $\alpha \in J^1$
$(N f)(i) = \overline{f(ni)}$.

Definition 2 *A filter of a lattice $(L, \vee, \wedge, 0, 1)$ is a nonempty subset F of L such that:*
  (i) If $x, y \in F$, then $x \wedge y \in F$
  (ii) If $x \in F$ and $x \leq y$, then $y \in F$.

A filter F of L is called proper filtre if it does not contains 0.

Definition 3 *Let L be a lattice and U a proper filter in L . Then U is said to be a maximal filter (more usually known as an ultrafilter) if the only filter properly containing U is L.*

We will need the following Theorem and Lemma.

Theorem 1 *([9], Theorem 2.1) Let B be a Boolean algebra and let X be the set of its ultrafilters. Then B is isomorphic to $\mathcal{P}(X)$, the embedding monomorphism being given by*

$$\sigma(x) = \{U \in X \ / \ x \in U\}.$$

Lemma 1 *([9], Lemma 3.2) Let $(L, J, (\varphi_\alpha)_{\alpha\in J^0}, (\Psi_\alpha)_{\alpha\in J^1}, N)$ be an involutive $\theta-$valued £ukasiewicz-Moisil algebra. Then*

$$(\Psi_\beta \geq \varphi_\alpha, \ for\ all \ \ 0 < \beta < \alpha < 1)$$

Definition 4 *Two involutive $\theta-$valued £ukasiewicz-Moisil algebras $(£, J, (\varphi_\alpha)_{\alpha\in J^0}, (\Psi_\alpha)_{\alpha\in J^1}, n, N)$ and $\left(£', J, (\varphi'_\alpha)_{\alpha\in J^0}, (\Psi'_\alpha)_{\alpha\in J^1}, n, N'\right)$ are said to be homomorphic if there exists a morphism $f : £ \longrightarrow £'$ such that:*
(i) $f \circ \varphi_\alpha = \varphi'_\alpha \circ f$, for all $\alpha \in J^0$;
(ii) $f \circ N = N' \circ f$.
If f is a monomorphism, we say that £ can be embedded into £'.

We note that (i) and (ii) implies $f \circ \Psi_\alpha = \Psi'_\alpha \circ f$,$\forall \alpha \in J^1$.

2.2 L-fuzzy sets

We consider $L-$fuzzy set as a mapping from a nonempty set X into a complete chain L with largest element $\top$ and least element $\bot$, provided with an unary, involutive, order-reversing operator $\tilde{n}$. In particular, L can be the $Int([0,1])$, where $Int([0,1])$ stands for the set of all closed subinterval of $[0,1]$. But will not be studied here. The class of $L-$fuzzy sets on a universe X will be denoted $F_L(X)$.

Further the set $F_L(X)$ can be equipped with a structure of involutive θ-valued £ukasiewicz-Moisil algebra $(F_L(X), L, (N_i)_{i\in L^\bot}, (N'_j)_{j\in L^\top}, \tilde{N}, \tilde{n})$ as follows:
for any $A, B \in F_L(X)$, $i \in L^\bot = L - \{\bot\}$, $j \in L^\top = L - \{\top\}$,

1) The structure $(F_L(X), \cup, \cap)$ of all $L-$fuzzy sets on a universe X is a distributive lattice with the least element $\emptyset$ defined by $\emptyset(x) = \bot$, for all $x \in X$ and the greatest element X. The union and the intersection are defined as follows:

$$A \cup B(x) = sup(A(x), B(x)), \qquad \text{for all } x \in X.$$
$$A \cap B(x) = inf(A(x), B(x)), \qquad \text{for all } x \in X.$$

2) $N_i : F_L(X) \longrightarrow \mathcal{P}(X)$ defined by:

$$N_i(A) = \{x \in X \ / \ A(x) \geq i\}$$

3) $N'_j : F_L(X) \longrightarrow \mathcal{P}(X)$ defined by:

$$N'_j(A) = \{x \in X \ / \ A(x) > j\}.$$

4) $\tilde{N}A(x) = \tilde{n}(A(x)), \qquad \text{for all} x \in X.$

3 Representation theorem

In this section, we will show under the condition that J is a complete chain, every involutive θ–valued £ukasiewicz-Moisil algebra can be embedded in an algebra of L–fuzzy sets. But first we prove two lemmas.

Lemma 2 *let J be a complete chain with an unary, involutive, order-reversing operator n. Then the pair $(L, \leq_L)$ defined by*

$$L = \{(\alpha, n\alpha) \ / \ \alpha \in J\},$$

$$(\alpha, n\alpha) \leq_L (\beta, n\beta) \Leftrightarrow \alpha \leq \beta$$

is a complete chain with an unary, involutive, order-reversing operator $\tilde{n}$ defined by

$$\tilde{n}(\alpha, n\alpha) = (n\alpha, \alpha), \qquad \textit{for all } \alpha \in J.$$

Proof. In [6, Lemma 2.1] it is proven that if J a complete lattice with an unary, involutive, order-reversing operator n, then the pair $(L, \leq_L)$ defined by

$$L = \{(\alpha, \beta) \in J \times J \ / \ \alpha \leq n\beta\},$$

$$(\alpha_1, \beta_1) \leq_L (\alpha_2, \beta_2) \Leftrightarrow \alpha_1 \leq \alpha_2 \wedge \beta_1 \geq \beta_2$$

is a complete lattice.

Now, consider that $\beta = n\alpha$ and J is a complete chain with an unary, involutive, order-reversing operator n. Hence, we may conclude that the pair $(L, \leq_L)$ defined by

$$L = \{(\alpha, \beta) \in J \times J \ / \ \alpha \leq n\beta\} = \{(\alpha, n\alpha) \ / \ \alpha \in J\},$$

$$(\alpha, n\alpha) \leq_L (\beta, n\beta) \Leftrightarrow \alpha \leq \beta$$

is a complete chain.

Moreover, since n is an unary, involutive, order-reversing operator in J we have that $\tilde{n}$ defined by

$$\tilde{n}(\alpha, n\alpha) = (n\alpha, \alpha), \qquad \text{for all } \alpha \in J$$

is an unary, involutive, order-reversing operator in L.

It is also evident from the definition of lattices isomorphism, that the mapping ℓ defined as

$$\begin{array}{rccc} \ell: & J & \longrightarrow & L \\ & \alpha & \longmapsto & (\alpha, n\alpha) \end{array}$$

is an isomorphism between the chains J and L. $\square$

Lemma 3 *Let $(\pounds, J, (\varphi_\alpha)_{\alpha\in J^0}, (\Psi_\alpha)_{\alpha\in J^1}, N)$ be an involutive $\theta-$valued £ukasiewicz-Moisil algebra. Then*

$$\sup\{\alpha \in J^0 \;/\; \varphi_\alpha(x) \in U\} = \sup\{\beta \in J^1 \;/\; \Psi_\beta(x) \in U\}.$$

Proof. For all $\alpha \in J^0$ and $\beta \in J^1$ we can distinguish at least two cases. In the one hand, if $\beta < \alpha$, then from the inequality ($\Psi_\beta \geq \varphi_\alpha,$ *for all* $0 < \beta < \alpha < 1$) given by Lemma1 it follows that $\varphi_\alpha(x) \in U$ implies $\Psi_\beta(x) \in U$. This means that

$$\sup\{\beta \in J^1 \;/\; \Psi_\beta(x) \in U\} \leq \sup\{\alpha \in J^0 \;/\; \varphi_\alpha(x) \in U\} \qquad (1)$$

In the other hand, if $\beta \geq \alpha$ then $\Psi_\beta \leq \Psi_\alpha \leq \varphi_\alpha$. Moreover, if $\Psi_\beta(x) \in U$ then $\varphi_\alpha(x) \in U$. This means that

$$\sup\{\beta \in J^1 \;/\; \Psi_\beta(x) \in U\} \geq \sup\{\alpha \in J^0 \;/\; \varphi_\alpha(x) \in U\} \qquad (2)$$

It is clear from (1) and (2) that:

$$\sup\{\alpha \in J^0 \;/\; \varphi_\alpha(x) \in U\} = \sup\{\beta \in J^1 \;/\; \Psi_\beta(x) \in U\}.$$

$\square$

Theorem 2 *The mapping $f : \pounds \longrightarrow F_L(X)$ defined, $\forall x \in \pounds$ and $\forall U \in X$ by:$f(x)(U) = (\lambda_x, n\lambda_x)$, where*

$$\lambda_x = \sup\{\alpha \in J^0 \;/\; \varphi_\alpha(x) \in U\},$$

is a monomorphism between the involutive $\theta-$valued £ukasiewicz-Moisil algebra $(\pounds, J, (\varphi_\alpha)_{\alpha\in J^0}, (\Psi_\alpha)_{\alpha\in J^1}, n, N)$ and $\left(F_L(X), L, (N_\alpha)_{\alpha\in L^\perp}, (N'_\alpha)_{\alpha\in L^\top}, \tilde{N}, \tilde{n}\right)$ the algebra of $L-$fuzzy sets, where J is a complete chain, X is the set of ultrafilters of $C(\pounds)$ and L is the complete chain with $\tilde{n}$ defined as in Lemma2.

Proof. Since J is a complete chain with n we obtain that $f(x)(U) \in L$, i.e., f is well defined.

First we shall show that f is a lattices morphism.

i) $f(0)(U) = (\lambda_0, n\lambda_0)$ such that:

$$\begin{aligned} \lambda_0 &= \sup\{\alpha \in J^0 \;/\; \varphi_\alpha(0) \in U\} \\ &= \sup\{\alpha \in J^0 \;/\; 0 \in U\} \\ &= \sup\emptyset = 0, \end{aligned}$$

Hence, $f(0)(U) = (0,1) = \bot$. Then we have

$$f(0) = \emptyset.$$

$f(1)(U) = (\lambda_1, n\lambda_1)$ such that:

$$\begin{aligned} \lambda_1 &= \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(1) \in U\} \\ &= \sup\{\alpha \in J^0 \ / \ 1 \in U\} \\ &= \sup J^0 = 1, \end{aligned}$$

Hence, $f(1)(U) = (1,0) = \top$. Then we have

$$f(1) = X.$$

ii) $f(x \vee y)(U) = (\lambda_{x\vee y}, n\lambda_{x\vee y})$ such that:

$$\begin{aligned} \lambda_{x\vee y} &= \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(x \vee y) \in U\} \\ &= \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(x) \vee \varphi_\alpha(y) \in U\} \\ &= \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(x) \in U \ \ \vee \varphi_\alpha(y) \in U\} \\ &= \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(x) \in U\} \vee \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(y) \in U\} \\ &= \lambda_x \vee \lambda_y, \end{aligned}$$

Hence,

$$\begin{aligned} f(x \vee y)(U) &= (\lambda_x \vee \lambda_y, n(\lambda_x \vee \lambda_y)) \\ &= (\lambda_x, n\lambda_x) \vee^* (\lambda_y, n\lambda_y) \\ &= sup(f(x)(U), f(y)(U)). \end{aligned}$$

Thus

$$f(x \vee y)(U) = f(x) \cup f(y)(U).$$

iii) Similarly,

$$f(x \wedge y)(U) = f(x) \cap f(y)(U).$$

Now we shall prove that $f \circ \varphi_\alpha = N_{(\alpha, n\alpha)} \circ f$, for all $\alpha \in J^0$ and $f \circ N = \tilde{N} \circ f$.

i)

$$\begin{aligned} f \circ \varphi_\alpha(x)(U) &= f(\varphi_\alpha(x))(U) \\ &= (\lambda_{\varphi_\alpha(x)}, n\lambda_{\varphi_\alpha(x)}) \end{aligned}$$

such that:

$$\begin{aligned}\lambda_{\varphi_\alpha(x)} &= \sup\{\beta \in J^0 \ / \ \varphi_\beta(\varphi_\alpha(x)) \in U\} \\ &= \sup\{\beta \in J^0 \ / \ \varphi_\alpha(x) \in U\} \\ &= \{ \begin{matrix} 0 & if \varphi_\alpha(x) \notin U \\ 1 & if \varphi_\alpha(x) \in U \end{matrix}\end{aligned}$$

Therefore

$$f \circ \varphi_\alpha(x) = \{ \begin{matrix} \emptyset & if \varphi_\alpha(x) \notin U \\ X & if \varphi_\alpha(x) \in U \end{matrix} = \sigma(\varphi_\alpha(x)).$$

In the other hand

$$\begin{aligned}N_{(\alpha,n\alpha)} \circ f(x) &= N_{(\alpha,n\alpha)}(f(x)) \\ &= \{U \in X \ / \ f(x)(U) \geq (\alpha, n\alpha)\} \\ &= \{U \in X \ / \ \alpha \in \{\beta \in J^0 \ / \ \varphi_\beta(x) \in U\}\} \\ &= \{U \in X \ / \ \varphi_\alpha(x) \in U\} \\ &= \sigma(\varphi_\alpha(x))\end{aligned}$$

Hence

$$f \circ \varphi_\alpha = N_{(\alpha,n\alpha)} \circ f, \qquad \text{for all } \alpha \in J^0.$$

2) We prove that $f \circ N = \tilde{N} \circ f$. However, before doing this we need to show that

$$\sup\{\alpha \in J^0 \ / \ \Psi_{n\alpha}(x) \notin U\} = n \sup\{\alpha \in J^1 \ / \ \Psi_\alpha(x) \in U\}. \tag{3}$$

Let us consider that

$$\sup\{\alpha \in J^0 \ / \ \Psi_{n\alpha}(x) \notin U\} \neq n \sup\{\alpha \in J^1 \ / \ \Psi_\alpha(x) \in U\},$$

then there exists $\alpha \in \{\alpha \in J^0 \ / \ \Psi_{n\alpha}(x) \notin U\}$ in which $\alpha < n\lambda$ or $\alpha > n\lambda$ where $\lambda = \sup\{\alpha \in J^1 \ / \ \Psi_\alpha(x) \in U\}$. Moreover, $n\alpha \geq \lambda$ or $n\alpha \leq \lambda$ implies that $\Psi_{n\alpha}(x) \leq \Psi_\lambda(x)$ or $\Psi_{n\alpha}(x) \geq \Psi_\lambda(x)$. Hence $\Psi_\lambda(x) \notin U$ or $\Psi_{n\alpha}(x) \in U$, this is contradiction with the hypothesis.

From the above equality (3) we obtain for all $x \in L$,

$$f \circ N(x) = f(Nx) = (\lambda_{Nx}, n\lambda_{Nx})$$

where:

$$\begin{aligned}\lambda_{Nx} &= \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(Nx) \in U\} \\ &= \sup\{\alpha \in J^0 \ / \ N\Psi_{n\alpha}(x) \in U\} \\ &= \sup\{\alpha \in J^0 \ / \ \Psi_{n\alpha}(x) \notin U\} \\ &= n \sup\{\alpha \in J^1 \ / \ \Psi_\alpha(x) \in U\}\end{aligned}$$

We know by Lemma 3 that

$$\sup\{\alpha \in J^1 \ / \ \Psi_\alpha(x) \in U\} = \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(x) \in U\}.$$

Then

$$\begin{aligned} \lambda_{Nx} &= n \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(x) \in U\} \\ &= n\lambda_x. \end{aligned}$$

Therefore

$$\begin{aligned} f \circ N(x) &= (n\lambda_x, \lambda_x) \\ &= \tilde{n}(\lambda_x, n\lambda_x) \\ &= \tilde{n}(f(x)) \\ &= \tilde{N} f(x) \end{aligned}$$

Hence

$$f \circ N = \tilde{N} \circ f.$$

Finally, f is injective since for $x, y \in \pounds$ holds $f(x) = f(y)$, then $\forall U \in X$, we have $f(x)(U) = f(y)(U)$.

Hence, $(\lambda_x, n\lambda_x) = (\lambda_y, n\lambda_y)$.

Then we have $\lambda_x = \lambda_y$,i.e.,

$$\sup\{\alpha \in J^0 \ / \ \varphi_\alpha(x) \in U\} = \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(y) \in U\},$$

which yields

$$\{\alpha \in J^0 \ / \ \varphi_\alpha(x) \in U\} = \{\alpha \in J^0 \ / \ \varphi_\alpha(y) \in U\}.$$

Then, for all $\alpha \in J^0$, we have $\varphi_\alpha(x) \in U$ iff $\varphi_\alpha(y) \in U$. Consequently

$$\{U \in X \ / \ \varphi_\alpha(x) \in U\} = \{U \in X \ / \ \varphi_\alpha(y) \in U\}.$$

Applying Theorem 1 yields $\varphi_\alpha(x) = \varphi_\alpha(y)$ for all $\alpha \in J^0$, and then $x = y$ (Moisil's determination principle).

This completes the proof of Theorem 2. $\square$

4 Conclusion

In this paper, we have found under the condition that J is a complete chain, every $\theta-$valued involutive £ukasiewicz-Moisil algebra $(\pounds, J, (\varphi_\alpha)_{\alpha\in J^0}, (\Psi_\alpha)_{\alpha\in J^1}, n, N)$ can be embedded in an algebra of $L-$fuzzy sets $\left(F_L(X), L, (N_\alpha)_{\alpha\in L^\perp}, (N'_\alpha)_{\alpha\in L^\top}, \tilde{N}, \tilde{n}\right)$ where L is a complete chain with $\tilde{n}$ depends on J has an order type $\tilde{\theta} = \theta$. In particular, if L is the $Int([0,1])$ where $Int([0,1])$ is the set of all closed subinterval of $[0,1]$, we can also obtain that $(\pounds, J, (\varphi_\alpha)_{\alpha\in J^0}, (\Psi_\alpha)_{\alpha\in J^1}, n, N)$ can be embedded in $(F_L(X), L, (N_\alpha)_{\alpha\in L^\perp}, (N'_\alpha)_{\alpha\in L^\top}, \tilde{N}, \tilde{n})$ and the monomorphism f can be defined by

$$f(x)(U) = [0, \lambda_x]$$

where:

$$\lambda_x = \sup\{\alpha \in J^0 \ / \ \varphi_\alpha(x) \in U\}.$$

References

[1] A. Amroune, Représentation des algebres de £ukasiewicz θ-valentes involutives par des structures floues, *BUSEFAL (Institut de Recherche en Informatique de Toulouse)*, 43(1990), 5-11.

[2] V. Boicescu, A. Filipoiu, G. Georgescu, and S. Rudeanu, *£ukasiewicz Moisil algebras*, North Holland, Elsevier, Amesterdam, New york, 1991.

[3] R. Cignoli, *Algebras de Moisil*, Notas de Logica Matematica, 27, Instituto de Matematica, Universidad del Sur, Bahia Blanca, 1970.

[4] J. Coulon and J. L. Coulon, A propos de la représentation des algebres de £ukasiewicz et des algebres Booleennes floues, *Rev. Roumaine Math. Pures et Appl.*, 34(1989), 403-411.

[5] J. Coulon and J. L. Coulon, Un nouveau résultat concernant la représentation d'une algebre de £ukasiewicz involutive dans l'algebre des parties floues d'une structure floue involutive, *Rev. Roumaine Math. Pures et Appl.*, 38(1993), 319-326.

[6] G. Deschrjver and E. E. Kerre, On the relationship between some extensions of fuzzy set theory, *Fuzzy Sets and Systems*, 133(2003), 227-235.

[7] D. Dubois, W. Ostasiewicz and H. Prade, Fuzzy sets: history and basic notation, in D. Dubois, H. Prade (Eds.), Fundamentals of Fuzzy Sets, *Kluwer Academic Publishers, Dordrecht*, 2000, 80-93.

[8] J. Goguen, L-fuzzy sets, *J. Math. Anal. Appl.* 18 (1967), 145-174.

[9] M. De Glas, Representation of Lukasiewicz many-valued algebras, *J. Math. Anal. Appl.*114 (1986), 315-327.

[10] GR. C. Moisil, Notes sur les logiques non-chrysippiennes, *Ann. Sci. Univ. Jassy* 27 (1941), 86-98. Reprinted in [13], 233-237.

[11] GR. C. Moisil, £ukasiewician algebras, Preprint, *competing Center, Univ. of Bucharest*, 1968. Reprinted in [13], 311-324.

[12] GR. C. Moisil, *Essais sur les logiques non chrysippiennes*, Editions de l'Academie, Bucarest 1972.

[13] GR. C. Moisil, *Ensembles flous et logiques plusieurs valeurs*, Université de Montréal, Mai 1973.

[14] GR. C. Moisil, Sur l'emploi des mathématiques dans les sciences de l'homme *Accd. Naz. Lincei. Contributi del Centro Linceo Interdisciplinare di Sci. Mat. eloro Appl.*, 17(1976).

[15] D. Ponasse, Algebres floues et algebres de £ukasiewicz, *Rev. Roumaine Math. Pures et Appl.*, 23(1978), 103-111.

[16] S. Rudeanu, On £ukasiewicz Moisil algebras of fuzzy sets, *Studia Logica*, 52 (1993), 95-111.

[17] Guo-Jun Wang and Ying-Yutte, Intuitionistic fuzzy sets and L-fuzzy sets, *Fuzzy Sets and Systems*, 110(2000), 271-274.

[18] L. A. Zadeh, Fuzzy sets, *Inform. and Control.*, 8 (1965), 338-353.

[19] H. -J. Zimmermann, *Fuzzy set theory and its applications*, Kluwer Academic Publishers, Boston, Dordrecht, London, 1991.

L. Zedam and A. Amroune
department of Mathematics, M'sila University,
P.O. Box 166 Ichbilia, M'sila 28105, Algeria

lzedam@yahoo.fr aamrounedz@yahoo.fr