Fuzzy Anti-Covered Left Ideals in Semigroups [1]

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Abstract: In this paper, we introduce the concept of fuzzy anti-covered left ideal of semigroups , some properties of fuzzy anti-covered left ideals of semigroups are proved. Furthermore, we introduce the notion of completely fuzzy left ideal and prove that if S is a regular semigroup, then every completely fuzzy left ideal of a semigroup S is constant.

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1. Introduction and preliminaries

Throughout this paper, S always presents a semigroup.

The ideals in semigroup have been introduced universally, since Zadeh introduced the notion of fuzzy set[1], many people have investigated fuzzy ideals in ring, semiring and semigroup, etc. In [4,5] Fabrici introduced covered left ideal, which means a left ideal L of S satisfying $L \subseteq S(S-L)$, and covered ideal, which is a two-sided ideal L of S satisfying $L \subseteq S(S-L)S$ respectively, and obtained some properties in terms of maximal ideals and bases of S. In [3], Xie investigated ordered semigroups containing no maximal ideals , also introduced covered ideals in ordered semigroups.

In this note, we define a new type of left ideal from the converse inclusion relation, introduce a fuzzy anti-covered left ideal and obtain some properties of fuzzy anti-covered left ideals from the view of fuzzy theory, we also study completely fuzzy left

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ideal.

Let L be a nonempty subset of a semigroup S, L of S is called a *left ideal* if $SL \subseteq L$. A proper left ideal L of S is called a *covered left ideal* (briefly CL-ideal) of S if $L \subseteq S(S-L)$[5].

For the necessary definitions, the reader is refered to [2].

2. Properties of fuzzy ACL-ideal

Definition 2.1 A nonempty subset L of S is called a *anti-covered left ideal* (briefly ACL-ideal) of S) if $S(S-L) \subseteq L$. L is called a *anti-covered right ideal* (briefly ACR-ideal) of S) if $(S-L)S \subseteq L$.

Both anti-covered left and anti-covered right ideal will be called a *completely anti-covered ideal* (briefly CAC-ideal) of S.

Definition 2.2 A subset L of S is called a *AC interior ideal* of S if $S(S-L)S \subseteq L$. L is called a AC-biideal if $(S-A)S(S-A) \subseteq A$.

Definition 2.3 A fuzzy subset f of S is called a *fuzzy ACL-ideal* if $S \circ (1-f) \subseteq f$, where $S(x) \equiv 1$ for any x in S.

Theorem 2.4 *Let f be a fuzzy subset of S. Then the followings are equivalent:*

(1) f is a fuzzy ACL-ideal of S.

(2) $(\forall x, y \in S)$ $f(xy) \geq 1 - f(y)$.

(3) $(\forall x \in S^2)$ $f(x) \geq 1 - \bigwedge_{x=yz} f(z)$.

Proof (1) $\Rightarrow$ (2). If f is a fuzzy ACL-ideal, then $S \circ (1-f) \subseteq f$. Thus $f(xy) \geq S \circ (1-f)(xy) \geq 1 - f(y)$.

(2) $\Rightarrow$ (3). Suppose that $(\forall x, y \in S)$ $f(xy) \geq 1-f(y)$. Then for any $x = yz \in S^2$, $f(x) = f(yz) \geq 1 - f(z)$. Since y, z are arbitrary, we have $f(x) \geq \bigvee_{x=yz} \{1 - f(z)\} = 1 - \bigwedge_{x=yz} f(z)$.

(3) $\Rightarrow$ (1). Suppose that $(\forall x \in S^2)\ \ f(x) \geq 1 - \bigwedge_{x=yz} f(z)$. Then for any x in S, if $x \neq yz$ for some y, z in S, we have $S \circ (1-f)(x) = 0 \leq f(x)$. If $x = yz \in S^2$ for some y, z in S, then

$$S \circ (1-f)(x) = \bigvee_{x=yz} \{S(y) \bigwedge (1-f)(z)\} = \bigvee_{x=yz} \{1 - f(z)\} = 1 - \bigwedge_{x=yz} f(z) \leq f(x),$$

that is $S \circ (1-f) \subseteq f$. Thus f is a fuzzy ACL-ideal of S. □

Definition 2.5 A fuzzy subset f of S is called a *fuzzy AC-interior ideal* if $S \circ (1-f) \circ S \subseteq f$. A fuzzy subset f of S is called a *fuzzy AC biideal* if $(1-f) \circ S \circ (1-f) \subseteq f$.

Theorem 2.6 *A fuzzy subset f of S is a fuzzy AC-interior ideal if and only if $f(xyz) \geq 1 - f(y)$ for any x, y, z in S.*

Proof If f is a fuzzy AC interior ideal, then $S \circ (1-f) \circ S \subseteq f$. Thus

$$f(xyz) \geq S \circ (1-f) \circ S(xyz) \geq 1 - f(y).$$

Conversely, suppose that $f(xyz) \geq 1 - f(y)$ for any x, y, z in S. Then for any x in S, if $S \circ (1-f) \circ S(x) = 0$, it is obvious that $S \circ (1-f) \circ S \subseteq f$. If $S \circ (1-f) \circ S(x) \neq 0$, then

$$\begin{aligned}
S \circ (1-f) \circ S(x) &= S \circ (1-f) \circ S(x) \\
&= \bigvee_{x=yz} \{S \circ (1-f)(y)\} \\
&= \bigvee_{x=yz} \{ \bigvee_{y=uv} \{S(u) \bigwedge (1-f)(v)\}\} \\
&= \bigvee_{x=yz} \{ \bigvee_{y=uv} \{1 - f(v)\}\} \\
&\leq \bigvee_{x=yz} \{ \bigvee_{y=uv} f(uvz)\} \\
&= \bigvee_{x=yz} \{f(yz)\} \\
&= f(x)
\end{aligned}$$

That is, $S \circ (1-f) \circ S \subseteq f$. □

Theorem 2.7 *A fuzzy subset f of S is a fuzzy AC biideal if and only if $f(xyz) \geq min\{1 - f(x), 1 - f(z)\}$, for any x, y, z in S.*

Proof If f is a fuzzy AC-biideal, then $(1-f)\circ S\circ(1-f)\subseteq f$. Thus $f(xyz)\geq (1-f)\circ S\circ(1-f)(xyz)\geq min\{1-f(x),1-f(z)\}$ for any x,y,z in S.

Conversely, suppose that $f(xyz)\geq \min\{1-f(x),1-f(z)\}$ for any x,y,z in S. Then for any x in S, if $(1-f)\circ S\circ(1-f)(x)=0$, it is clear that $(1-f)\circ S\circ(1-f)\subseteq f$. Otherwise,

$$\begin{aligned}
(1-f)\circ S\circ(1-f)(x) &= \bigvee_{x=yz}\{(1-f)\circ S(y)\bigwedge(1-f)(z)\} \\
&= \bigvee_{x=yz}\{\bigvee_{y=uv}\{(1-f)(u)\bigwedge S(v)\}\bigwedge(1-f(z)\} \\
&= \bigvee_{x=yz}\{\bigvee_{y=uv}\{1-f(u)\}\bigwedge\{1-f(z)\}\} \\
&= \bigvee_{x=yz}\{\bigvee_{y=uv}\{(1-f(u))\bigwedge(1-f(z))\}\} \\
&\leq \bigvee_{x=yz}\{\bigvee_{y=uv} f(uvz)\} \\
&= \bigvee_{x=yz}\{f(yz)\} \\
&= f(x).
\end{aligned}$$

That is $(1-f)\circ S\circ(1-f)\subseteq f$. □

Theorem 2.8 *L is a ACL-ideal of S if and only if f_L is a fuzzy ACL-ideal of S .*

Proof Let L be a ACL-ideal of S. Then $S(S-L)\subseteq L$. For any x,y in S if $y\in L$, then $1-f_L(y)=1-1=0\leq f_L(xy)$. If $y\in S-L$, then $xy\in S(S-L)\subseteq L$, $f_L(xy)=1\geq 1-f_L(y)=1$. It follows that $f_L(xy)\geq 1-f_L(y)$.

Conversely, if f_L is a fuzzy ACL-ideal of S, then $f_L(xy)\geq 1-f_L(y)$. For $\forall x\in S, y\in S-A$, $f_L(xy)\geq 1-f_L(y)=1-0=1$, and so $f_A(xy)=1$, it follows that $xy\in A$, that is to say $S(S-A)\subseteq A$. □

In generally, f is a fuzzy ACL-ideal of S, then f_t need not to be ACL-ideal of S, we illustrate it by following example.

Example 2.9 Let $S=\{a,b,c,d\}$ with the multiplication table:

.	a	b	c	d
a	a	a	a	a
b	a	b	c	b
c	a	b	c	c
d	a	b	c	a

Define fuzzy subset f of S as follows, $f(a) = 2/3, f(b) = 1/2, f(c) = 3/5, f(d) = 1/3$. It is easy to verify that f is a fuzzy ACL-ideal. But $A = f_{3/5} = \{a, c\}$ is not an ACL-ideal. In fact, $S(S - A) = S\{b, d\} = \{a, b, c\} \not\subseteq A$.

If we let $t \leq 1/2$, we have the following theorem.

Theorem 2.10 *Let f be a fuzzy ACL-ideal of S and $t \in [0, 1/2]$, then every proper subset of S $f_t(\neq \emptyset)$ is a ACL-ideal of S.*

Proof If f is a fuzzy ACL-ideal of S, then $f(xy) \geq 1 - f(y)$ for any x, y in S. We need to show that $S(S - f_t) \subseteq f_t$ foe $t \leq 1/2$. For $\forall x \in S, y \in S - f_t$, we have $f(y) < t$, then $f(xy) \geq 1 - f(y) > 1 - t \geq t$ by $t \leq 1/2$, and so $xy \in f_t$. Thus $S(S - f_t) \subseteq f_t$. □

The converse of Theorem 2.10 is not true in general.

Example 2.11 Let $S = \{a, b, c, d\}$ with the multiplication table:

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	b

Define a fuzzy subset f of S as follows:

$$f(a) = 1/2, f(b) = 2/3, f(c) = 1/4, f(d) = 3/5.$$

For all $t \in [0, 1/2]$, $f_t = f_{1/2}$ or $f_t = f_{1/4} = S$, it is easy to verify that $A = f_{1/2} = \{a, b, c\}$ is a ACL-ideal. Since $S(S - A) = Sd = \{a, b\} \subseteq A$. But f is not a fuzzy ACL-ideal. In fact, $f(ac) = f(a) = 1/2 < 3/4 = 1 - f(c)$.

Theorem 2.12 *Let f_1 and f_2 be two fuzzy subsets of S and $f_1 \subseteq f_2$. If f_1 is a fuzzy ACL-ideal, then f_2 is also a fuzzy ACL-ideal.*

Proof f_1 is a fuzzy ACL-ideal and $f_1 \subseteq f_2$ implies that $S \circ (1-f_2) \subseteq S \circ (1-f_1) \subseteq f_1 \subseteq f_2$, and so f_2 is also a fuzzy ACL-ideal of S. □

Theorem 2.13 *If f_1 and f_2 are two fuzzy ACL- ideals of S, then $f_1 \cup f_2$ is a fuzzy ACL-ideal.*

Proof Let f_1 and f_2 be two fuzzy ACL-ideals of S. Then $S \circ (1-f_1) \subseteq f_1, S \circ (1-f_2) \subseteq f_2$. Thus $S \circ (1 - f_1 \cup f_2) \subseteq S \circ (1 - f_1) \subseteq f_1 \subseteq f_1 \cup f_2$. □

If f_1 and f_2 are two fuzzy ACL- ideals of S, $f_1 \cap f_2$ is not a fuzzy ACL-ideal of S in general. We have a following example:

Example 2.14 Let $S = \{a, b, c, d\}$ with the multiplication table:

.	a	b	c	d
a	a	a	a	a
b	a	b	c	c
c	a	c	c	d
d	a	c	c	c

Define fuzzy subsets f_1 and f_2 as follows, $f_1(a) = 4/5, f_1(b) = 3/4, f_1(c) = 2/3, f_1(d) = 1/3, f_2(a) = 7/8, f_2(b) = 6/7, f_2(c) = 5/6, f_2(d) = 1/4$.

It is easy to verify that f_1 and f_2 are both fuzzy ACL-ideals of S. But $f_1 \cap f_2$ is not. In fact: $(f_1 \cap f_2)(a) = 4/5, (f_1 \cap f_2)(b) = 3/4, (f_1 \cap f_2)(c) = 2/3, (f_1 \cap f_2)(d) = 1/4, (f_1 \cap f_2)(cd) = (f_1 \cap f_2)(c) = 2/3 < 3/4 = 1 - ((f_1 \cap f_2)(d))$.

Both fuzzy left and fuzzy ACL-ideal f of S is called a *completely left ideal* of S.

Theorem 2.15 Let f be a fuzzy subset of S. Then the followings are equivalent:

(1) f is a completely fuzzy left ideal of S.

(2) $(\forall x, y \in S)\ f(xy) \geq \max\{f(y), 1 - f(y)\}$.

(3) $(\forall x \in S^2) f(x) \geq \max\{\bigvee_{x=yz} f(z), \bigvee_{x=yz} (1 - f(z))\}$.

Proof (1) $\Rightarrow$ (2). Since f is a completely fuzzy left ideal of S implies that $f(xy) \geq f(y)$ and $f(xy) \geq 1 - f(y)$, we have $f(xy) \geq \max\{f(y), 1 - f(y)\}$.

(2) $\Rightarrow$ (3). Let $f(xy) \geq \max\{f(y), 1 - f(y)\}$ for any x, y in S. Then for any x in S^2, there exists y, z in S such that $x = yz$, $f(x) = f(yz) \geq f(z)$, $f(x) = f(yz) \geq 1 - f(z)$. Thus $f(x) \geq \bigvee_{x=yz} f(z)$, $f(x) \geq \bigvee_{x=yz} \{1 - f(z)\}$, and so $f(x) \geq \max\{\bigvee_{x=yz} f(z), \bigvee_{x=yz} \{1 - f(z)\}\}$.

(3) $\Rightarrow$ (1). Let $f(x) \geq max\{\bigvee_{x=yz} f(z), \bigvee_{x=yz} \{1-f(z)\}\}$ for any x in S^2. Then if $x \notin S^2$, it is clear that $0 = S \circ f(x) \leq f(x)$, $0 = S \circ (1-f)(x) \leq f(x)$. If $x \in S^2$,

$$\begin{aligned} S \circ f(x) &= \bigvee_{x=yz} \{S(y) \bigwedge f(z)\} \\ &= \bigvee_{x=yz} \{f(z)\} \leq f(x), \\ S \circ (1-f)(x) &= \bigvee_{x=yz} \{S(y) \bigwedge \{1-f(z)\}\} \\ &= \bigvee_{x=yz} \{1-f(z)\} \leq f(x). \end{aligned}$$

Thus we conclude that f is a completely fuzzy left ideal. □

From the above (3), we can see that if f is a completely fuzzy left ideal, then $f(x) \geq 1/2$ for any $x \in S^2$.

Theorem 2.16 *L is a completely fuzzy left ideal of S if and only if f_L is a completely fuzzy left ideal.*

Proof Let L be ACL-ideal. Then f_L is left ideal. By theorem 2.8, f_L is also a fuzzy ACL-ideal, and so f_L is a completely fuzzy left ideal.

Conversely, it is clear. □

Theorem 2.17 Let f be a completely fuzzy left ideal of S. Then for the proper subset of S $f_t(\neq \emptyset)$ is a completely left ideal for any $t \in [0, 1/2]$.

Definition 2.18 [2] S is a *fuzzy left duo (fuzzy right duo)* if its every fuzzy left ideal (fuzzy right ideal) is also fuzzy right ideal (fuzzy left ideal). S is called *fuzzy duo* if it is fuzzy left duo and fuzzy right duo.

Definition 1.19 S is a *fuzzy AC-left duo (fuzzy AC-right duo)* if its every fuzzy ACL- ideal (fuzzy ACR-ideal) is also fuzzy ACR-ideal (fuzzy ACL-ideal). S is called *fuzzy AC-duo* if it is fuzzy AC-left duo and fuzzy AC-right duo. S is completely fuzzy duo if it is completely fuzzy left duo and completely fuzzy right duo.

From this point of view, we have

Theorem 2.20 *If S is commutative, then S is completely fuzzy duo.*

Theorem 2.21 *If S is regular, then every completely fuzzy left ideal f of S is constant.*

Proof If S is regular, then $(\forall x \in S)(\exists a \in S)\ x = xax \in S^2$. For a completely fuzzy left ideal f, $f(x) \geq 1/2$. In the other hand, $f(x) = f(xax) \geq 1 - f(x)$, then $f(x) \leq 1/2$, and so we conclude $f(x) = 1/2$ for any x in S. □

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