Countable α-Compactness, α-Metacompactness and the Fuzzy Topological Game $G`(DK, X)$

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Abstract

An attempt is made to interpret cluster points and accumulation points in Fuzzy Topological Spaces (fts) using the concept of α–shadings and a Characterisation of Countable Compactness in an fts is obtained. Further some close relationships of countable α-compactness with α– metacompactness and the Fuzzy Topological Game $G`(DK, X)$ are also investigated.

Keywords:- Fuzzy Topology , Metacompactness, Countable Compactness, Fuzzy Topological Game

1.Introduction

The concept of fuzzy covers was introduced by Chang[1]. The notion of a shading family was introduced in literature by Gantner and others[3] during the investigation of compactness in fts . The shading families are a very natural generaliation of coverings

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In particular 1^*- shading family of fuzzy sets is a fuzzy covering in the sense of Chang[1].

Now most of the properties of countably compact spaces in general topology are discussed in terms of cluster points and accumulation points. So we define α-cluster points and α–accumulation points in fts in a language which is closely related to shading families and in this framework we obtain a characterisation for countable compactness in fts.

The author[5] has introduced metacompactness in fts through α–shading families. Using this a necessary and sufficient condition for a space to be α–compact is obtained. Again some relations of countably α–compact spaces and fuzzy topological Game $G`(DK, X)$ introduced by the author [6] are also investigated.

2.Preliminaries

2.1 Definition[3] Let (X,T) be an fts and $\alpha \in [0,1]$. A collection $\boldsymbol{U}$ of fuzzy sets is called an α-shading (resp. α^*- shading) of X if for each $x \in X$, there exists $g \in \boldsymbol{U}$ with $g(x) > \alpha$ (resp. $g(x) \geq \alpha$).

2.2 Definition[3] A subcollection of an α-shading of X which is also an α-shading (resp. α^*- shading) is called an α-subshading (resp. α^*- subshading) of X.

The study of compactness in fuzzy topology was initiated by Chang[1]. Since then different compactness notions have been defined. The following notion is due to Gantner, Steinlage, and Warren[3].

2.3 Definition [3] A fts X is said to be α-compact (resp. α^*- compact) if each α- shading (resp. α^*- shading) of X by open fuzzy sets has a finite α-subshading (resp. α^*- subshading) . Where $\alpha \in [0,1]$.

2.4 Definition[4]. A fts X is said to be countably α-compact (resp.countably α^*-compact) if every countable α- shading (resp. α^*- shading) of X by open fuzzy sets has a finite α-sub shading (resp. α^*- sub shading). Where $\alpha \in [0,1]$.

2.5 Definition[4].A fts X is said to be α -Lindelof (resp. α^*- Lindelof) if every α-shading (resp. α^*- shading) of X by open fuzzy sets has a countable α-subshading (resp. α^*- subshading). Where $\alpha \in [0,1]$.

2.6 Definition. Let $\alpha \in [0,1]$. An α-cluster point (resp. α^*- cluster point) of a set A in a fts X is a fuzzy point x_λ such that each fuzzy nbd U of x_λ with $U(x) > \alpha$ (resp. $U(x) \geq \alpha$) contains some fuzzy point of A with distinct support.

2.7 Definition. A sequence $(x^n_{\lambda n})$ of fuzzy points with distinct support in a fuzzy topological space X α-accumulates at x_λ (resp. α^*-accumulates) at x_λ if and only if for every fuzzy nbd U of x_λ with $U(x) > \alpha$ (resp. $U(x) \geq \alpha$) and for every $n \in N$, there is an $m \geq n$ such that $x^m_{\lambda m} < U$ and (λ_n)accumulates at λ in the crisp sense in $[0,1]$

2.8 Theorem . The following are equivalent in a fuzzy topological Space.

(i) X is countably α - compact.

(ii) Every fuzzy subset of X with countably infinite support has at least one α - cluster point

(iii) Every sequence of fuzzy points in X with distinct support has an α -accumulation point.

Proof:

(i) $\Rightarrow$ (ii)

If possible let A be a fuzzy subset of X with countably infinite support and has no α-cluster point. Then it follows that every fuzzy point $x^i_{\lambda i}$ in A has a fuzzy nbd Ui with

$Ui\ (x^i) > \alpha$ which contains no other fuzzy point of A with distinct support. Now $Supp(A)$ clearly closed and $X \setminus Supp(A)$ is open . Now consider the collection

$\chi_{X \setminus Supp(A)} \cup \{U_i : i \in N\}$.This is clearly a countable α-shading of X by open fuzzy sets which has no finite α-subshading.

(ii) $\Rightarrow$ (iii)

Let $(x^n_{\lambda n})$ be a sequence of fuzzy points in X with distinct support. Then there are two possibilities.

(a) Cardinality of the support of the range set is countably infinite. Then by (ii) this has atleast one α-cluster point say x_λ .Now every fuzzy nbd U of x_λ with $U(x)>\alpha$ contains infinitely many points of the sequence other than x_λ. Clearly this x_λ is an α-accumulation point of the sequence. For, For any $n \in N$ the set $\{x^n_{\lambda n} : 1 \leq n \leq N\}$ is finite .There fore it follows that for any nbd U of x_λ with $U(x)>\alpha$ and for any $n \in N$, there is an $m \geq n$ such that $x^m_{\lambda m} < U$ and (λ_n) accumulates at λ.

(b) If cardinality of range set is finite , then there should be some fuzzy point x_λ with $x^n_{\lambda n} = x_\lambda$ for infinitely many $n \in N$. Then clearly this x_λ is an α-accumulation point.

(iii) $\Rightarrow$ (i)

Let X be not countably α-compact . Let $U=\{U_1, U_2, U_3 \dots \dots\}$ be a countable α-shading of X by open fuzzy sets which has no finite α- subshading. Therefore $\{U_1, U_2, U_3 \dots U_k\}$ cannot α-shade X for any finite k. Therefore corresponding to each finite k we can find an $x^k \in X$ such that $U_j(x^k) > \alpha$ for some $j>k$ and $U_i(x^k) \leq \alpha$ for $1 \leq i \leq k$.

Let $U_j(x^k) = \eta_k$ where $\eta_k \in (\alpha, 1]$. Now the sequence $(x^k_{\eta k})$ has no α-accumulation point. For, if possible let x_η be an α-accumulation point of $(x^k_{\eta k})$. Now since U is an α-shading of X, we can find a minimum $l \in N$ such that $U_l(x) > \alpha$ and $U_i(x) \leq \alpha$ for all $1 \leq i \leq l$. Now take $n = l+1$ and consider the nbd U_l of x. Then for any $m \geq n$ we have $x^m_{\eta m} > U_l$. For corresponding to any m, we can find some U_j such that $U_j(x^m) > \alpha$ for some $j > m$.and $U_i(x^m) \leq \alpha$ for $1 \leq i \leq m$. Here $m \geq n = l+1$. Therefore $l < m$ and it follows that $U_l(x^m) \leq \alpha$ .But $\eta_m \in (\alpha, 1]$. Thus $x^m_{\eta m} < U_l$ which is a contridiction. This completes the proof.

3. Irreducible and Removable Shading Families.

3.1 Definition. Let $\mathcal{U}$ be an α-shading of a fts X. We say that $\mathcal{U}$ is irreducible if when a single member is removed from $\mathcal{U}$ then $\mathcal{U}$ is no longer an α-shading of X.

3.2 Definition. Let $\mathcal{U}$ be an α--shading of a fts X. A family $F \subset \mathcal{U}$ is removable if when we remove the collection F from $\mathcal{U}$, then also $\mathcal{U}$ is an α-shading of X.

3.3 Definitions [4] A family $\{a_s : s \in S\}$ of fuzzy sets in a fuzzy topological space (X,T) is said to be locally finite if for each x in X there exists an open fuzzy set g of X with $g(x) = 1$ such that $a_s \leq 1-g$ holds for all but atmost finitely many s in S.

3.4 Definition. [4] A family $\{a_s : s \in S\}$ of fuzzy sets in a fts (X,T) is said to be point finite if for each x in X, $a_s(x) = 0$ for all but atmost finitely many s in S. Or equivalently as $a_s(x) > 0$ for atmost finitely many s in S.

3.5 Definition [4] Let (X,T) be an fts and $\alpha \in [0,1)$. Let $\mathcal{U}$ and $\mathcal{V}$ be any two α-shadings (respα*-shading) of X. Then $\mathcal{U}$ is a refinement of $\mathcal{V}$ ($\mathcal{U} < \mathcal{V}$) if for each $g \in \mathcal{U}$ there is an $h \in \mathcal{V}$ such that $g \leq h$.

3.6 Definition [4] A fts (X,T) is said to be α-paracompact (resp. α *- paracompact) if each α-shading (resp. α*- shading) of X by open fuzzy sets has a locally finite α-shading (resp. α*-shading) refinement by open fuzzy sets. Where $\alpha \in [0,1]$.

3.7 Definition[5]. A fuzzy topological space (X, T) is said to be α-metacompact (resp. α* -metacompact) if each α-shading (resp. α*- shading) of X by open fuzzy sets has a point finite α-shading.(resp. α*- shading) refinement by open fuzzy sets. Where $\alpha \in [0,1]$.

3.8 Lemma. Let (X,T) be a fts. Then for every point finite α-shading of X, there is an irreducible α-subshading of X.

Proof:

Let $\mathcal{U}$ be a point finite α-shading of X . Consider the set $\mathcal{R}$ of all removable subcollections of $\mathcal{U}$. Partial order $\mathcal{R}$ by inclusion. For any chain $\{R_\mu\}$ in $\mathcal{R}$, there is an upper bound say $R = \bigcup_\mu R_\mu$. Now clearly $R \in \mathcal{R}$. Otherwise there would be some $x \in X$ such that R contains finitely many $U_1.U_2,....U_n$ with $U_i(x) > \alpha$ for $1 \le i \le n$. Since $\{R_\mu\}$ is a chain all the U_i's belong to some R_μ say $R_{\mu'}$. Which will contradict $R_{\mu'} \in \mathcal{R}$. Hence by Zorn's lemma , there is a maximal $R_o \in \mathcal{R}$ and so $\mathcal{U} \setminus R_o$ is irreducible.

3.9 Theorem. A fts X is α-compact if and only if it is both countably α-compact and α-metacompact.

Proof:

Necessary follows clearly. For sufficiency part , Let X be countably α-compact and α-metacompact. Let $\boldsymbol{U} = \{u_\alpha : \alpha \in A\}$ be any α-shading of X by open fuzzy sets . Now since X is α-metacompact it follows that $\boldsymbol{U}$ has a point finite α-shading refinement by open fuzzy sets , say $\{v_\beta : \beta \in B\}$. Now by lemma 3.8 $\{v_\beta : \beta \in B\}$ has an irreducible subshading say $\{v_\gamma : \gamma \in G\}$. Now this should be finite. For since $\{v_\gamma : \gamma \in G\}$ is irreducible , corresponding to each v_γ we can find an $x^\gamma \in X$ with $v_\gamma(x^\gamma) > \alpha$ and $v_k(x^\gamma) \le \alpha$

for every $k \neq \gamma$. Now let $v_\gamma(x^\gamma) = \eta_\gamma$ where $\eta_\gamma \in (\alpha, 1]$. If $\{v_\gamma : \gamma \in G\}$ were infinite , then$\{ x^\gamma_{\eta\gamma} : \gamma \in G \}$ would be an infinite fuzzy subset with no α-cluster points. Contradicting X is countably α-compact.

Now corresponding to each v_γ choose some $u_{k(\gamma)} > v_\gamma$. This is possible since $\{v_\gamma : \gamma \in G\}$ is a refinement of $\{u_\alpha : \alpha \in A\}$. Thus we can reduce $\{u_\alpha : \alpha \in A\}$ to a finite subfamily. Thus X is α-compact and the proof is complete.

As an immediate consequence of Theorem 3.9 and from the Definition of α-Lindelof spaces we get the following corollary

3.10 Corollary. Countable α-compactness is equivalent to compactness in α-paracompact spaces and in arbitrary α-Lindelof spaces.

4.Countably α-compact spaces and the Fuzzy topological Game G`(DK,X)

As a generalisation of the Topological Game G(***K***,*X*)introduced by Telgarsky [10] the author [6] introduced the Fuzzy Topological Game G`(***K***,*X*) . Where ***K*** is a non empty family of fuzzy topological spaces, where all spaces are assumed to be T_1. That is all fuzzy singletons are fuzzy closed . $\underline{I}^x$ denote the family of all fuzzy closed subsets of X. Also $X \in$ ***K*** implies $\underline{I}^x \subseteq$ ***K*** . ***DK*** denote the class of all fuzzy topological spaces which have a discrete fuzzy closed α-shading by members of ***K***.

4.1 Definition[4].A family $\{a_s : s \in S\}$ of fuzzy sets in a fuzzy topological space (X,T) is said to be discrete if for each x in X , there exists an open fuzzy set g of X with $g(x) = 1$ such that $a_s \leq 1 - g$ holds for all but atmost one s in S.

4. 2 Definition [6] Let ***K*** be a class of fuzzy topological spaces and let $X \in$ ***K*** . Then the fuzzy topological game *G`(**K**,X)* is defined as follows. There are two players Player *I* and

Player II . They alternatively choose consecutive terms of the sequence$(E_1,F_1,E_2,F_2,...)$of fuzzy subsets of X . When each player chooses his term he knows $\boldsymbol{K}$, X and their previous choices. A sequence $(E_1,F_1,E_2,F_2,...)$ is a play for $G^{`}(\boldsymbol{K},X)$ if it satisfies the following conditions for each $n \geq 1$.

(1) E_n is a choice of Player I

(2) F_n is a choice of Player II

(3) $E_n \in \underline{I}^X \wedge \boldsymbol{K}$

(4) $F_n \in \underline{I}^X$

(5) $E_n \vee F_n < F_{n-1}$ where $F_0 = X$

(6) $E_n \wedge F_n = 0$

Player I wins the play if $\inf_{n\geq 1} F_n = 0$. Otherwise Player II wins the Game.

4.3 Definition [6] A finite sequence $(E_1,F_1,E_2,F_2,........,E_m,F_m)$ is admissible if it satisfies conditions (1) -- (6) for each $n \leq m$.

4.4 Definition [6] Let S` be a crisp function defined as follows

$$S^{`}: \bigcup_{n \geq 1} (\underline{I}^X)^n \xrightarrow{\text{into}} \underline{I}^X \cap \boldsymbol{K}$$

Let $S_1=\{x\}$

$S_2=\{F\in \underline{I}^X : (S^{`}(X),F)$ is admissible for $G^{`}(\boldsymbol{K},X)$ }. Continuing like this inductively we get $S_n=\{(F_1,F_2,F_3,.......Fn)$: $(E_1,F_1,E_2,F_2,...E_n,F_n)$is admissible for $G^{`}(\boldsymbol{K},X)$ where $F_0=X$ and $Ei=S^{`}(E_1,F_1,E_2,F_2.....F_{i-1})$for each $i \leq n\}$. Then the restriction S of $S^{`}$ to $\bigcup_{n\geq 1} S_n$ is called a fuzzy strategy for Player I in $G^{`}(\boldsymbol{K},X)$.

4.5 Definition [6] If Player I wins every play $(E_1,F_1,E_2,F_2,...E_n,F_n,.............)$ such that $E_n = S(F_1,F_2,.........,F_{n-1})$, then we say that S is a fuzzy winning strategy.

4.6 Definition [6] $S: \underline{I}^X \xrightarrow{into} \underline{I}^X \cap \boldsymbol{K}$ is called a fuzzy stationery strategy for Player I in $G^{`}(\boldsymbol{K},X)$ if $S(F) < F$ for each $F \in \underline{I}^X$ .We say that S is a fuzzy stationary winning strategy if he wins every play ($S(X),F_1,S(F_1),F_2.........$).

From definitions above, we get

4.7 Result [6] A function $S: \underline{I}^X \xrightarrow{into} \underline{I}^X \cap \boldsymbol{K}$ is a fuzzy stationary winning strategy if and only if it satisfies

(i) For each $F \in \underline{I}^X$, $S(F) < F$

(ii) If $\{F_n: n \geq 1\}$ satisfies $S(X) \wedge F_1 = 0$ and $S(F_n) \wedge F_{n+1} = 0$ for each $n \geq 1$ then

$$\underset{n\geq 1}{Inf}\, F_n = 0 .$$

4.8 Theorem[6] Player I has a fuzzy winning strategy in $G^{`}(\boldsymbol{K},X)$ if and only if he has a fuzzy stationary winning strategy in it.

4.9 Definition. A collection $\{A_i : i \in I\}$ of subsets of a fuzzy topological space X is said to be closure preserving if for each $J \subseteq I$, cl $[\vee A_i : i \in J]$ = $\vee_{i \in J}$ cl $[A_i]$

4.10 Theorem. If $\boldsymbol{C}$ is a closure preserving α-shading of a fts X by fuzzy closed and countably α-compact sets and if $\boldsymbol{K}$ is a class of fts with $\boldsymbol{C} \subset \boldsymbol{K}$, then Player I has a fuzzy stationary winning strategy in $G^{`}(D\boldsymbol{K},X)$

Proof:

Corresponding to each fuzzy closed set F in X , consider the collection$\{C \wedge F: C \in \boldsymbol{C}\}$ and let $\boldsymbol{D}(F)$ be the maximal disjoint subcollection of this. This is possible since

C is an α-shading of X. Clearly $\mathcal{D}(F)$ is closure preserving and disjoint and hence it is discrete. Now define $S: \underline{I}^x \xrightarrow{\text{into}} \underline{I}^x \cap DK$ by $F \mapsto \vee \mathcal{D}(F)$. We will show that S is a fuzzy stationary winning strategy for Player I in G`(DK,X).

Let $\{F_n : n \in N\}$ be a decreasing $(F_1 > F_2 > F_3 \ldots)$ sequence with $S(X) \wedge F_1 = 0$ and $S(F_n) \wedge F_{n+1} = 0$. If possible let if $\underset{n\geq 1}{Inf}\, F_n \neq 0$. Then there exists $C_0 \in \boldsymbol{C}$ such that C_0 has non empty meet with each of F_n. Now $C_0 \wedge F_n \notin \mathcal{D}(F_n)$ for each $n \geq 1$. For, If $C_0 \wedge F_n \in \mathcal{D}(F_n)$ for some n,then

$$C_0 \wedge F_n = (C_0 \wedge F_n) \wedge F_{n+1}$$

$$< [\vee \mathcal{D}(F_n)] \wedge F_{n+1}$$

$$= S(F_n) \wedge F_{n+1}$$

$= 0$. This is a contradiction. Therefore $C_0 \wedge F_n \notin \mathcal{D}(F_n)$ for each for every $n \geq 1$.

Fix some $n \geq 1$. $\mathcal{D}(F_n)$ is maximal and disjoint. Also $C_0 \wedge F_n \notin \mathcal{D}(F_n)$. Therefore we can take some $C_n \in \boldsymbol{C}$ such that $C_n \wedge F_n \in \mathcal{D}(F_n)$ and $(C_n \wedge C_0) \wedge F_n \neq 0$. For each $n \geq 1$, take some $x^n \in X$ such that $[(C_0 \wedge F_n) \wedge C_n]\ (x^n) > \alpha$ where $\alpha \in (0,1]$. Let Min $\{C_0(x), F_n(x), C_n(x) = \lambda_n$.Now clearly we have$[S(F_n)](x^n) > \alpha$..Also$S(F_n) \wedge F_{n+1} = 0$. Therefore $F_{n+1}(x^n) = 0$. Now consider the sequence $(x^n{}_{\lambda n})$ in C_0. Now C_0 is countably α-compact. There fore it has an α-cluster point say x_λ in C_0.This follows from Theorem2.8.

Now we have $\underset{n\geq 1}{Inf}\, F_n(x) > \alpha$ .For, if $F_n(x) \leq \alpha$ for some n, then we can choose some $m \geq n$ with $\lambda_m > F_n(x^m)$. But $F_m < F_n$. Therefore $F_m(x^m) < F_n(x^m)$. Now $\lambda_m \leq F_m(x^m) <$ $F_n(x^m)$.There fore $\lambda_m < F_n(x^m)$. This is a contradiction.

Now claim $Sup_{n\geq 1} C_n(x) = 0$. For , let $C_n(x) > 0$ for some n . Now $C_0 \wedge F_n \in \mathcal{D}(F_n)$ and $F_{n+1}(x) > \alpha$ .Then $(C_n \wedge F_n \wedge F_{n+1})(x) < (S(Fn) \wedge F_{n+1})(x)$

$= 0$. There fore $C_n(x) = 0$. This is a contradiction.

Since C is closure preserving, we have $cl\{x^n_{\lambda n} : n\geq 1\}(x) > \alpha$. Also $cl\{x^n_{\lambda n} : n\geq 1\} < cl\ Sup_{n\geq \bar{1}} C_n = Sup_{n\geq \bar{1}} C_n$. There fore $Sup_{n\geq \bar{1}} C_n(x) > \alpha$, where $\alpha \in (0,1]$. This is a contradiction to $Sup_{n\geq \bar{1}} C_n(x) = 0$. This completes the proof.

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