Properties of Fuzzy Total Continuity *

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Abstract: *In this paper, some characteristic properties and basic properties of fuzzy totally continuous and fuzzy totally semi-continuous functions are established, and a fuzzy totally open function is introduced. By the way we'd like to point out that a theorem and two counterexamples are incorrect in a paper by A.Mukherjee.*

Keywords:*Fuzzy topology; Fuzzy clopen set; Fuzzy totally continuous function*

1. Introduction and Preliminaries

Continuity and its stronger forms constitute an important area in the field of general topology. These notions have been proved to be of fundamental importance in the realm of fuzzy topology. Mukherjee introduced the fuzzy totally continuous and fuzzy totally semi-continuous functions[4] in fuzzy topological spaces in the Chang's[3] sense. But he did not study their characteristic properties. The main purpose of this paper is to study some characteristic properties and basic properties of fuzzy totally continuous and fuzzy totally semi-continuous functions, and introduce fuzzy totally open functions. By the way we'd like to point out that Theorem 2.6, Example 2.5 and Example 2.7 are incorrect in [4].

Throughout the paper by (X, δ) and (Y, τ) or simply bby X and Y we mean fuzzy topological spaces[3] (fts, for short). For a fuzzy set A in X, the notations A^o, A^-, A_o, A_- and A' will respectively stand for the interior, closure, semi-interior, semi-closure complement of A.

Definition 1.1. A function $f : (X, \delta) \to (Y, \tau)$ from fts (X, δ) to another fts (Y, τ) is called:

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(1) Fuzzy totally continuous if $f^{-1}(B)$ is a fuzzy clopen set of X for each $B \in \tau$[4].

(2) Fuzzy totally semi-continuous if $f^{-1}(B)$ is a fuzzy semi-clopen set of X for each $B \in \tau$[4].

(3) Fuzzy completely continuous if $f^{-1}(B)$ is a fuzzy regular set of X for each $B \in \tau$[2].

2. Properties of Fuzzy Total Continuity

Theorem 2.1. Let $f : (X, \delta) \to (Y, \tau)$ be a function. Then the following are equivalent:

(1) f is fuzzy totally continuous.

(2) $f^{-1}(B)$ is a fuzzy clopen set of X for each $B \in \tau'$.

(3) $(f^{-1}(B))^{-} \leq f^{-1}(B^{-})$ and $f^{-1}(B) \leq (f^{-1}(B^{-}))^{o}$ for each subset B of Y.

(4) $f^{-1}(B^{o}) \leq (f^{-1}(B))^{o}$ and $(f^{-1}(B^{o}))^{-} \leq f^{-1}(B)$ for each subset B of Y.

Proof. (1) $\Rightarrow$ (2) : Let $B \in \tau'$. Then $B' \in \tau$ and from (1), $f^{-1}(B')$ is fuzzy clopen. From $f^{-1}(B') \in \delta$ i.e., $(f^{-1}(B))' \in \delta$, we have $f^{-1}(B) \in \delta'$. From $f^{-1}(B') \in \delta'$ i.e., $(f^{-1}(B))' \in \delta'$, we have $f^{-1}(B) \in \delta$. Thus, $f^{-1}(B)$ is fuzzy clopen.

(2) $\Rightarrow$ (3) : Let B be a fuzzy set in Y. Then $B^{-} \in \tau'$. By (2), $f^{-1}(B^{-}) \in \delta'$, hence,

$(f^{-1}(B))^{-} \leq (f^{-1}(B^{-}))^{-} = f^{-1}(B^{-})$.

Again by (2), $f^{-1}(B^{-}) \in \delta$, hence,

$f^{-1}(B) \leq f^{-1}(B^{-}) = (f^{-1}(B^{-}))^{o}$.

(3) $\Rightarrow$ (4) : Let B be a fuzzy set in Y. By first formula of (3), we have

$f^{-1}(B'^{-}) \geq (f^{-1}(B'))^{-} = (f^{-1}(B))'^{-}$.

Hence,

$f^{-1}(B^{o}) = f^{-1}(B'^{-\prime}) = (f^{-1}(B'^{-}))' \leq (f^{-1}(B))'^{-\prime} = (f^{-1}(B))^{o}$.

By second formula of (3), we have

$f^{-1}(B') \leq (f^{-1}(B'^{-}))^{o} = (f^{-1}(B^{o\prime}))^{o} = (f^{-1}(B^{o}))'^{o}$.

Hence,

$f^{-1}(B) = (f^{-1}(B'))' \geq (f^{-1}(B^{o}))'^{o\prime} = (f^{-1}(B^{o}))^{-}$.

(4) $\Rightarrow$ (1) : Let $B \in \tau$. Then $B = B^{o}$. By first formula of (4), we have

$f^{-1}(B) = f^{-1}(B^{o}) \leq (f^{-1}(B))^{o}$.

Hence, $f^{-1}(B) = (f^{-1}(B))^{o}$, i.e., $f^{-1}(B) \in \delta$. By second formula of (4), we have

$f^{-1}(B) \geq (f^{-1}(B^{o}))^{-} = (f^{-1}(B))^{-}$.

Hence, $f^{-1}(B) = (f^{-1}(B))^{-}$, i.e., $f^{-1}(B) \in \delta'$. Thus, f is fuzzy totally continuous.

Theorem 2.2. Let $f : (X, \delta) \to (Y, \tau)$ be fuzzy totally continuous, x_α be a fuzzy point in X. If for each $B \in \tau$ and $f(x_\alpha) \in B$, then there is a fuzzy clopen set A of X such that $x_\alpha \in A$ and $f(A) \leq B$.

Proof. Let $B \in \tau$ and $f(x_\alpha) \in B$. Then $x_\alpha \in f^{-1}(B)$. Let $A = f^{-1}(B)$. Since f fuzzy totally continuous, A is fuzzy clopen of X. Hence, $f(A) = ff^{-1}(B) \leq B$.

Theorem 2.3. If $f : X_1 \to X_2$ is fuzzy totally continuous and $g : X_2 \to X_3$ is fuzzy continuous, then $g \circ f$ is fuzzy totally continuous.

Theorem 2.4. Let $p_i : X_1 \times X_2 \to X_i (i = 1, 2)$. be the projection of $X_1 \times X_2$ on X_i. If $f : X \to X_1 \times X_2$ is fuzzy totally continuous, then $p_i \circ f$ is also fuzzy totally continuous.

Proof. This follows directly from Theorem 2.3.

Theorem 2.5. Let $f : X_1 \to X_2$ be a function. If the graph $g : X_1 \to X_1 \times X_2$ of f is fuzzy totally continuous, then f is also fuzzy totally continuous.

Proof. This follows directly from Theorem 2.4.

3. Properties of Fuzzy Total Semi-continuity

Theorem 3.1. Let $f : (X, \delta) \to (Y, \tau)$ be a function. Then the following are equivalent:

(1) f is fuzzy totally semi-continuous.
(2) $f^{-1}(B)$ is a fuzzy semi-clopen set of X for each $B \in \tau'$.
(3) $(f^{-1}(B))_- \leq f^{-1}(B^-)$ and $f^{-1}(B) \leq (f^{-1}(B^-))_o$ for each subset B of Y.
(4) $f^{-1}(B^o) \leq (f^{-1}(B))_o$ and $(f^{-1}(B^o))_- \leq f^{-1}(B)$ for each subset B of Y.

Proof. This is analogous to the proof of Theorem 2.1.

Theorem 3.2. Let $p_i : X_1 \times X_2 \to X_i (i = 1, 2)$. be the projection of $X_1 \times X_2$ on X_i. If $f : X \to X_1 \times X_2$ is fuzzy totally semi-continuous, then $p_i \circ f$ is also fuzzy totally semi-continuous.

Proof. This follows directly from Theorem 2.8 in [4].

Theorem 3.3. Let $f : X_1 \to X_2$ be a function. If the graph $g : X_1 \to X_1 \times X_2$ of f is fuzzy totally semi-continuous, then f is also fuzzy totally semi-continuous.

Proof. This follows directly from Theorem 3.2.

Now we point out that Theorem 2.6 is incorrect in [4]. Theorem 2.6 is:

Let (X, δ) be a fuzzy extremally disconnected space. If $f : (X, \delta) \to (Y, \tau)$ is fuzzy continuous (resp. fuzzy semi-continuous) then it is also fuzzy totally continuous (resp. fuzzy totally semi-continuous).

In fact, a fuzzy continuous function $f : (X, \delta) \to (Y, \tau)$ from a fuzzy extremally disconnected space (X, δ) to another fuzzy topological space (Y, τ) need not be fuzzy totally semi-continuous.

Example 3.4. Let $X = I$ and A be a fuzzy set of X defined as follows:

$$A(x) = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{2}, \\ x, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Clearly, $\delta = \{0, A, 1\}$ is a fuzzy topology on X and (X, δ) is a fuzzy extremally disconnected space. Consider the identity mapping $f : (X, \delta) \to (X, \delta)$. Then f is fuzzy continuous. Clearly, $f^{-1}(A) = A$ is not a fuzzy semiclosed set. Thus, f is not fuzzy totally semi-continuous.

Now we give the following theorem. Firstly we show a lemma.

Lemma 3.5. Let X be a fuzzy extremally disconnected space. If A is a fuzzy regular open set in X, then A is also a fuzzy closed set in X.
Proof. Let A be fuzzy regular open in X, then $A = A^{-o}$. Since X be fuzzy extremally disconnected, A^- is open in X, i.e., $A^{-o} = A^-$. Hence, $A = A^-$, i.e., A is closed in X.

Theorem 3.6. Let (X, δ) be a fuzzy extremally disconnected space. If $f : (X, \delta) \to (Y, \tau)$ is fuzzy completely continuous, then it is also fuzzy totally continuous.

Proof. Let $B \in \tau$, then $f^{-1}(B)$ is fuzzy regular open in (X, δ). Since (X, δ) is fuzzy extremally disconnected, $f^{-1}(B)$ is fuzzy closed in (X, δ) by Lemma 3.5. Clearly, $f^{-1}(B) \in \delta$. Hence, f is also fuzzy totally continuous.

In [4], Examples 2.5 and 2.7 are not right either. $\delta = \{0, A, A', 1\}$ is mistaken for a fuzzy topology on $X = I = [0, 1]$, where

$$A(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ 2x-1, & \frac{1}{2} \le x \le 1. \end{cases}$$

Clearly, $A \cup A' \notin \delta$ and $A \cap A' \notin \delta$. Hence, δ is not a fuzzy topology on I.

4. Fuzzy Totally Open Functions

Definition 4.1. A function $f : (X, \delta) \to (Y, \tau)$ from fts (X, δ) to another fts (Y, τ) is said to be fuzzy totally open if the image of every fuzzy open set of X is fuzzy clopen of Y.

Clearly, a fuzzy totally open function is fuzzy open. That the converse need not be true is shown by the following example.

Example 4.2. The f is described just as in Example 3.4. Clearly, f is fuzzy open. But f is not fuzzy totally open.

Theorem 4.3. A function $f : (X, \delta) \to (Y, \tau)$ is fuzzy open iff $f(A^o) \le (f(A))^o$ and $(f(A^o))^- \le f(A)$ for each fuzzy set A of X.

Proof. Let $f : (X, \delta) \to (Y, \tau)$ be a fuzzy totally open function and A be a fuzzy set in X. Then $A^o \in \delta$ and $f(A^o) \in \tau$. Hence,

$f(A^o) = (f(A^o))^o \le (f(A))^o$.

Again by $f(A^o) \in \tau'$. Hence, $(f(A^o))^- = f(A^o) \le f(A)$.

Conversely, let $A \in \delta$. Then $A = A^o$. By $f(A^o) \le (f(A))^o$, we have

$f(A) = f(A^o) \le (f(A))^o$.

Hence, $f(A) = (f(A))^o$, i.e., $f(A) \in \tau$. By $(f(A^o))^- \le f(A)$ we have

$f(A) \ge (f(A^o))^- = (f(A))^-$.

Hence, $f(A) = (f(A))^-$, i.e., $f(A) \in \tau'$. Thus, f is fuzzy totally open.

Theorem 4.4. If $f : X_1 \to X_2$ is fuzzy open and $g : X_2 \to X_3$ is fuzzy totally open, then $g \circ f$ is fuzzy totally open.

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