# Intuitionistic L-fuzzy Groups

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**Abstract:** The purpose of this paper is to construct the basic concepts of intuitionistic L-fuzzy groups. The concept of an intuitionistic L-fuzzy set, which is a generalization of the concept of a L-fuzzy set, has been introduced by Atanassov and Stoeva. In this paper, first, the fundamental definitions of intuitionistic L-fuzzy sets are given. Then, the definition of an intuitionistic L-fuzzy group is introduced and its simple properties are discussed. Finally, in the sense of homomorphism and isomorphism between two classical groups, the image and preimage of intuitionistic L-fuzzy groups are studied.

**Keywords:** intuitionistic L-fuzzy set; intuitionistic L-fuzzy group; homomorphism; isomorphism; image; preimage.

## 1.Introduction

With the research of fuzzy sets, Atanassov presented intuitionistic fuzzy sets which are very effective to deal with vagueness. On basis of this, Turanli and Coker introduced several types fuzzy connectedness in intuitionistic fuzzy topological spaces [3]. Bustince presented different theorems for building intuitionistic fuzzy relations [4]. Szmidt and Kacprzyk proposed a non-probabilistic-type entropy measure for intuitionistic fuzzy sets [5]. However, the algebraic aspects in intuitionistic fuzzy theory have gained less attention. In this study, we shall give a brief introduction to the intuitionistic L-fuzzy groups.

## 2.Preliminaries

Here we shall present the fundamental definitions. For the sake of completeness we shall outline the basic facts:

**Definition** 2. 1. [8] Let $(L,\vee,\wedge)$ is a complete lattice such that for all $A \subseteq L$ and for all $b$

$\in L$, $\vee\{a \wedge b \mid a \in A\} = (\vee\{a \mid a \in A\}) \wedge b$ and $\wedge\{a \vee b \mid a \in A\} = (\wedge\{a \mid a \in A\}) \vee b$.

The meet, join, and partial ordering of $L$ will be written as $\vee$, $\wedge$, and $\leq$, respectively. We also write 1 and 0 for the maximal and minimal elements of $L$, respectively.

**Definition** 2. 2. [2] Let $X$ be a nonempty set. An **intuitionistic** $L$**-fuzzy set** $A$ is an object having the form $A = \{\langle x, \overline{A}(x), \underline{A}(x)\rangle \mid x \in X\}$, where $\overline{A}: X \to L$ and $\underline{A}: X \to L$ are two functions.

**Definition** 2. 3. [2] Let $A$, $B$, and $A_j$ be intuitionistic $L$-fuzzy sets in a nonempty set $X$, $j \in J$.

(a) $A \subseteq B \iff \overline{A}(x) \leq \overline{B}(x)$ and $\underline{A}(x) \geq \underline{B}(x)$ for each $x \in X$;

(b) $A = B \iff A \subseteq B$ and $A \supseteq B$;

(c) $A \cup B = \{\langle x, \overline{A}(x) \vee \overline{B}(x), \underline{A}(x) \wedge \underline{B}(x)\rangle | x \in X\}$;

(d) $A \cap B = \{\langle x, \overline{A}(x) \wedge \overline{B}(x), \underline{A}(x) \vee \underline{B}(x)\rangle | x \in X\}$;

(e) $\cup_{j \in J} A_j = \{\langle x, \vee_{j \in J} \overline{A_j}(x), \wedge_{j \in J} \underline{A_j}(x)\rangle | x \in X\}$;

(f) $\cap_{j \in J} A_j = \{\langle x, \wedge_{j \in J} \overline{A_j}(x), \vee_{j \in J} \underline{A_j}(x)\rangle | x \in X\}$;

**Definition** 2.4. Let $A$ an intuitionistic $L$-fuzzy set in a nonempty set $X$. For $\alpha \in L$, define $\overline{A}_\alpha, \underline{A}_\alpha$ as follows:

$$\overline{A}_\alpha = \{x \mid x \in X, \overline{A}(x) \geq \alpha\}, \quad \underline{A}_\alpha = \{x \mid x \in X, \underline{A}(x) \leq \alpha\}$$

It is easy to verify that for intuitionistic $L$-fuzzy sets $A$ and $B$

(a) $A \subseteq B, \alpha \in L \Rightarrow \overline{A}_\alpha \subseteq \overline{B}_\alpha, \underline{A}_\alpha \supseteq \underline{B}_\alpha$;

(b) $\alpha \leq \beta, \alpha, \beta \in L \Rightarrow \overline{A}_\alpha \supseteq \overline{A}_\beta, \underline{A}_\alpha \subseteq \underline{A}_\beta$;

(c) $A = B \Leftrightarrow \overline{A}_\alpha = \overline{B}_\alpha, \underline{A}_\alpha = \underline{B}_\alpha, \alpha \in L$.

Now we shall define the preimage and image of intuitionistic $L$-fuzzy sets. Let $X$, $Y$ be two nonempty sets and $f: X \to Y$ a mapping.

**Definition** 2.5. (a) If $B$ an intuitionistic $L$-fuzzy set in a nonempty set $Y$, then the **preimage** of $B$ under $f$, denoted by $f^{-1}(B)$, is the intuitionistic $L$-fuzzy set in $X$ defined by

$$f^{-1}(B) = \{\langle x, \overline{B}(f(x)), \underline{B}(f(x))\rangle | x \in X\}$$

(b) If $A$ an intuitionistic $L$-fuzzy set in a nonempty set $X$, then the **image** of $A$ under $f$, denoted by $f(A) = \{\langle y, \overline{f(A)}(y), \underline{f(A)}(y)\rangle | y \in Y\}$, is the intuitionistic $L$-fuzzy set in $Y$ defined by

$$\overline{f(A)}(y) = \begin{cases} \vee\{\overline{A}(x) \mid f(x) = y, x \in X\}, & f(y) \neq \phi \\ 0, & f(y) = \phi \end{cases}$$

$$\underline{f(A)}(y) = \begin{cases} \wedge\{\underline{A}(x) \mid f(x) = y, x \in X\}, & f(y) \neq \phi \\ 1, & f(y) = \phi \end{cases}$$

## 3. Intuitionistic $L$-fuzzy groups

Now we generalize the concept of "$L$-fuzzy group" to intuitionistic $L$-fuzzy sets.

**Definition** 3.1. An **intuitionistic $L$-fuzzy group** (ILG for short) $A$ on a group $U$ is an intuitionistic $L$-fuzzy set in $U$ satisfying the following conditions:

(a) $\overline{A}(xy) \geq \overline{A}(x) \wedge \overline{A}(y)$, $\underline{A}(xy) \leq \underline{A}(x) \vee \underline{A}(y)$, for all $x, y \in U$

(b) $\overline{A}(x^{-1}) \geq \overline{A}(x)$, $\underline{A}(x^{-1}) \leq \underline{A}(x)$, for all $x \in U$

Clearly, $\overline{A}(x^{-1}) \geq \overline{A}(x) \Leftrightarrow \overline{A}(x^{-1}) \leq \overline{A}(x) \Leftrightarrow \overline{A}(x^{-1}) = \overline{A}(x)$, and $\underline{A}(x^{-1}) \leq \underline{A}(x) \Leftrightarrow \underline{A}(x^{-1}) \geq \underline{A}(x) \Leftrightarrow \underline{A}(x^{-1}) = \underline{A}(x)$.

**Proposition** 3. 2. Let $A$ be an intuitionistic $L$-fuzzy set on a group $U$, then $A$ is an ILG on $U$ if and only if $\overline{A}_\alpha$ and $\underline{A}_\alpha$ are subgroups of $U$ for all $\alpha \in L$.

**Proof.** Suppose $A$ is an ILG on $U$. Let $\alpha \in L$, $x, y \in \overline{A}_\alpha$. Then $\overline{A}(x) \geq \alpha$, and $\overline{A}(y) \geq \alpha$. Since $A$ is an ILG, $\overline{A}(xy^{-1}) \geq \overline{A}(x) \wedge \overline{A}(y^{-1}) = \overline{A}(x) \wedge \overline{A}(y) = \alpha$. Therefore, $xy^{-1} \in \overline{A}_\alpha$, so $\overline{A}_\alpha$ is a subgroup of $U$. Similarly, $\underline{A}_\alpha$ is a subgroup of $U$.

Conversely, suppose $\overline{A}_\alpha$ is a subgroup of $U$ for any $\alpha \in L$. Let $x, y \in U$, $\alpha = \overline{A}(x) \wedge \overline{A}(y)$, $\beta = \overline{A}(x)$. Since $\overline{A}_\alpha$ and $\overline{A}_\beta$ are subgroups of $U$, $xy \in \overline{A}_\alpha$, $x^{-1} \in \overline{A}_\beta$. Thus $\overline{A}(xy) \geq \alpha = \overline{A}(x) \wedge \overline{A}(y), \overline{A}(x^{-1}) \geq \beta = \overline{A}(x)$. Similarly, we have $\underline{A}(xy) \leq \underline{A}(x) \vee \underline{A}(y)$ $\underline{A}(x^{-1}) \leq \underline{A}(x)$. Hence, it follows that $A$ is an ILG on $U$.

**Proposition** 3. 3. Let $A$ and $B$ be ILGs on a group $U$, then $A \cap B$ is an ILG on $U$.

**Proof.** Let $A = \{\langle x, \overline{A}(x), \underline{A}(x)\rangle | x \in U\}$ and $B = \{\langle x, \overline{B}(x), \underline{B}(x)\rangle | x \in U\}$. Then we have

$$\{\langle x, \overline{A \cap B}(x), \underline{A \cap B}(x)\rangle | x \in U\} = \{\langle x, \overline{A}(x) \wedge \overline{B}(x), \underline{A}(x) \vee \underline{B}(x)\rangle | x \in U\}$$

For all $x, y \in U$,

$$\begin{aligned}
\overline{A \cap B}(xy) &= \overline{A}(xy) \wedge \overline{B}(xy) \\
&\geq (\overline{A}(x) \wedge \overline{A}(y)) \wedge (\overline{B}(x) \wedge \overline{B}(y)) \\
&= (\overline{A}(x) \wedge \overline{B}(x)) \wedge (\overline{A}(y) \wedge \overline{B}(y)) \\
&= \overline{A \cap B}(x) \wedge \overline{A \cap B}(y)
\end{aligned}$$

Similarly, $\underline{A \cap B}(xy) \leq \underline{A \cap B}(x) \vee \underline{A \cap B}(y)$. Also, we have

$$\overline{A \cap B}(x^{-1}) = \overline{A}(x^{-1}) \wedge \overline{B}(x^{-1}) \geq \overline{A}(x) \wedge \overline{B}(x)$$

$$\underline{A \cap B}(x^{-1}) = \underline{A}(x^{-1}) \vee \underline{B}(x^{-1}) \le \underline{A}(x) \vee \underline{B}(x)$$

Hence, $A \cap B$ is an ILG on $U$.

**Corollary** 3. 3. Let $A_j$ be an ILG on a group $U$, $j \in J$, then $\cap_{j \in J} A_j$ is an ILG on $U$.

Let $A$ and $B$ be ILGs on a group $U$, but it is uncertain to verify that $A \cup B$ is an ILG on $U$.

## 4.Homomorphism and isomorphism of intuitionistic $L$-fuzzy groups

In this section, we shall introduce the concepts of homomorphism and isomorphism of ILGs, and use them to develop results concerning ILGs.

**Proposition** 4. 1. Let $U_1$ and $U_2$ be groups. Let $f$ be a homomorphism of $U_1$ onto $U_2$, $A$ an ILG on $U_1$. Then $f(A)$ is an ILG on $U_2$.

**Proof.** $f(A) = \left\{ \left\langle y, \overline{f(A)}(y), \underline{f(A)}(y) \right\rangle \middle| y \in U_2 \right\}$. Now suppose there exist $y_1, y_2 \in U_2$, such that

$$\overline{f(A)}(y_1 y_2) < \overline{f(A)}(y_1) \wedge \overline{f(A)}(y_2)$$

or

$$\underline{f(A)}(y_1 y_2) > \underline{f(A)}(y) \vee \underline{f(A)}(y_2).$$

By definition 2. 5., considering $f^{-1}(y_1) \ne \phi, f^{-1}(y_2) \ne \phi$, we have

$$\overline{f(A)}(y_1 y_2) < (\vee\{\overline{A}(x) \mid f(x) = y_1, x \in U_1\}) \wedge (\vee\{\overline{A}(x) \mid f(x) = y_2, x \in U_1\})$$

or $$\underline{f(A)}(y_1 y_2) > (\wedge\{\underline{A}(x) \mid f(x) = y_1, x \in U_1\}) \vee (\wedge\{\underline{A}(x) \mid f(x) = y_2, x \in U_1\}).$$

If the former equality holds, there exist $x_1, x_2 \in U_1$ such that $y_1 = f(x_1), y_2 = f(x_2)$, and $\overline{f(A)}(y_1 y_2) < \overline{A}(x_1) \wedge \overline{(A)}(x_2)$. Since $f(x_1 x_2) = y_1 y_2$, $\overline{A}(x_1 x_2) \le \vee\{\overline{A}(x) \mid f(x) = y_1 y_2\} = \overline{f(A)}(y_1 y_2) < \overline{A}(x_1) \wedge \overline{(A)}(x_2)$. This contradicts ILG's definition. If the latter equality holds, the proofs are similar to those of the former. Consequently, for all $y_1, y_2 \in U_2$,

$$\overline{f(A)}(y_1 y_2) \ge \overline{f(A)}(y_1) \wedge \overline{f(A)}(y_2)$$

and

$$\underline{f(A)}(y_1 y_2) \le \underline{f(A)}(y) \vee \underline{f(A)}(y_2).$$

For any $y \in U_2$, $\overline{f(A)}(y^{-1}) = \vee\{\overline{A}(x) \mid f(x) = y^{-1}, x \in U_1\} = \vee\{\overline{A}(x) \mid f(x^{-1}) = y, x \in U_1\} = \vee\{\overline{A}(x^{-1})^{-1} \mid f(x^{-1}) = y, (x^{-1})^{-1} \in U_1\} = \vee\{\overline{A}(z^{-1}) \mid f(z) = y, z^{-1} \in U_1\} \ge \vee\{\overline{A}(z) \mid$

$f(z)=y, z^{-1}\in U_1\}=\vee\{\overline{A}(z)\mid f(z)=y, z\in U_1\}=\overline{f(A)}(y)$. For $\underline{f(A)}(y^{-1})$, the proof follows in a similar manner as above. Hence, $f(A)$ is an ILG on $U_2$.

**Corollary** 4. 2. Let $U_1$ and $U_2$ be groups, $f$ a homomorphism of $U_1$ onto $U_2$. Let $A_j$ be an ILG on $U_1$, $j\in J$, then $f(\cap_{j\in J} A_j)$ is an ILG on $U_2$.

**Proposition** 4. 3. Let $U_1$ and $U_2$ be groups. Let $f$ be a homomorphism of $U_1$ into $U_2$, $A$ an ILG on $U_2$. Then $f^{-1}(A)$ is an ILG on $U_1$.

**Proof.** By definition 2. 5., $\overline{f^{-1}(A)}(x)=\overline{A}(f(x))$, and $\underline{f^{-1}(A)}(x)=\underline{A}(f(x))$, $x\in U_1$. Let $x_1, x_2\in U_1$,

$$\overline{f^{-1}(A)}(x_1x_2)=\overline{A}(f(x_1x_2))=\overline{A}(f(x_1)f(x_2))$$

$$\geq\overline{A}(f(x_1))\wedge\overline{A}(f(x_2))=\overline{f^{-1}(A)}(x_1)\wedge\overline{f^{-1}(A)}(x_2)$$

And, for any $x\in U_1$,

$$\overline{f^{-1}(A)}(x^{-1})=\overline{A}(f(x^{-1}))=\overline{A}((f(x))^{-1})\geq\overline{A}(f(x))=\overline{f^{-1}(A)}(x)$$

Similarly, we have

$$\underline{f^{-1}(A)}(x_1x_2)\leq\underline{f^{-1}(A)}(x_1)\vee\underline{f^{-1}(A)}(x_2)$$

$$\underline{f^{-1}(A)}(x^{-1})\leq\underline{f^{-1}(A)}(x)$$

Hence, $f^{-1}(A)$ is an ILG on $U_1$.

**Corollary** 4. 4. Let $U_1$ and $U_2$ be groups, $f$ a homomorphism of $U_1$ into $U_2$. Let $A_j$ be an ILG on $U_2$, $j\in J$, then $f^{-1}(\cap_{j\in J} A_j)$ is an ILG on $U_1$.

**Proposition** 4. 5. Let $U_1$ and $U_2$ be groups. Let $f$ be an isomorphism of $U_1$ onto $U_2$, $A$ an ILG on $U_1$. Then $f^{-1}(f(A))=A$.

**Corollary** 4. 6. Let $U_1$ and $U_2$ be groups, $f$ an isomorphism of $U_1$ onto $U_2$. $A$ an ILG on $U_2$. Then $f(f^{-1}(A))=A$.

**Corollary** 4. 7. Let $U$ is a group, $f$ an automorphism of $U$. $A$ an ILG on $U$. Then

$$f(A)=A \Leftrightarrow f^{-1}(A)=A.$$

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