WEAKLY PREOPEN AND WEAKLY PRECLOSED FUNCTIONS IN FUZZY TOPOLOGY *

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ABSTRACT : In this paper, we introduce and characterize fuzzy weakly preopen and fuzzy weakly preclosed functions between fuzzy topological spaces and also study these functions in relation to some other types of already known functions.

1 Introduction and Preliminaries

The concept of fuzzy sets was introduced by Prof. L.A. Zadeh in his classical paper [19]. After the discovery of the fuzzy subsets, much attention has been paid to generalize the basic concepts of classical topology in fuzzy setting and thus a modern theory of fuzzy topology is developed. The notion of fuzzy subsets naturally plays a very significant role in the study of fuzzy topology which was introduced by C.L. Chang [4] in 1968. In 1980, Ming and Ming [9], introduced the concepts of quasi-coincidence and q-neighbourhoods by which the extensions of functions in fuzzy setting can very interestingly and effectively be carried out. In 1985, D.A. Rose [16] defined weakly open functions in topological spaces. In 1997 J.H. Park, Y.B. Park and J.S. Park [13] introduced the notion of weakly open functions in between fuzzy topological spaces. In this paper we introduce and discuss the concept of fuzzy weakly preopen functions which is weaker than fuzzy preopen and f.a.o.N functions

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introduced by [3] and Nanda [11] respectively and we obtained several characterizations and properties of these functions .We also study these functions comparing with other types of already known functions. Here is seen that fuzzy preopenness implies fuzzy weakly preopenness but not conversely. But under a certain condition the converse is also true. We also introduce and study the concept of fuzzy weakly preclosed functions.

Throughout this paper by (X, τ) or simply by X we mean a fuzzy topological space (fts, shorty) due to Chang [4]. A fuzzy point in X with support $x \in X$ and value p $(0 < p \leq 1)$ is denoted by x_p. Two fuzzy sets λ and β are said to be quasi-coincident (q-coincident, shorty) denoted by $\lambda q \beta$, if there exists $x \in X$ such that $\lambda(x) + \beta(x) > 1$ [9] and by $\bar{q}$ we denote "is not q-coincident". It is known [9] that $\lambda \leq \beta$ if and only if $\lambda \bar{q} (1 - \beta)$. A fuzzy set λ is said to be q-neighbourhood (q-nbd) of x_p if there is a fuzzy open set μ such that $x_p q \mu$.

The interior, closure, and the complement of a fuzzy set $\lambda \in X$ are denoted by $Int(\lambda), Cl(\lambda)$ and $1 - \lambda$ respectively. For definitions and results not explained in this paper, the reader is referred to [1,4,8,10,14,16,19] assuming them to be well known.

DEFINITIONS 1.1. A fuzzy set λ in a fts X is called,
(1) Fuzzy preopen [3] if $\lambda \leq Int(Cl(\lambda))$.
(2) Fuzzy preclosed [3] if $Cl(Int(\lambda)) \leq \lambda$.
(3) Fuzzy regular open [1] if $\lambda = Int(Cl(\lambda))$.
(4) Fuzzy regular closed [1] if $\lambda = Cl(Int(\lambda))$.
(5) Fuzzy α-open [3] if $\lambda \leq Int(Cl(Int(\lambda)))$.
(6) Fuzzy α-closed [3] if $Cl(Int(Cl(\lambda))) \leq \lambda$.

Recall that if, λ be a fuzzy set in a fts X then $pCl(\lambda) = \bigcap\{\beta : \beta \geq \lambda, \quad \beta$ is fuzzy preclosed$\}$ (resp. $pInt(\lambda) = \bigcup\{\beta : \lambda \geq \beta, \quad \beta$ is fuzzy preopen$\}$) is called a fuzzy preclosure of λ (resp. fuzzy preinterior of λ) [3].

RESULT 1.2. (1) A fuzzy set λ in a fts X is fuzzy preclosed (resp. fuzzy preopen) if and only if $\lambda = pCl(\lambda)$ (resp. $\lambda = pInt(\lambda)$) [3].
(2) Let λ a fuzzy set of a fts (X, τ). Then ;
(i) $Int(\lambda) \leq pInt(\lambda) \leq \lambda \leq pCl(\lambda) \leq Cl(\lambda)$.
(ii) $1_X - pInt(\lambda) = pCl(1_X - \lambda)$ and $pInt(1_X - \lambda) = 1 - pCl(\lambda)$.

DEFINITION 1.3. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a function from a fts (X, τ_1) into a fts (X, τ_2). The function f is called:
(i) fuzzy preclosed [3] if $f(\lambda)$ is a fuzzy preclosed subset of Y for each fuzzy closed set λ in X.
(ii) fuzzy open [4] if $f(\lambda)$ is a fuzzy open subset of Y for each fuzzy open subset λ of X.
(iii) fuzzy weakly open [13] if $f(\lambda) \leq Int(f(Cl(\lambda)))$ for each fuzzy open set λ in X.
(iv) fuzzy almost open (written as f.a.o.N) [11] if $f(\lambda)$ is a fuzzy open subset of Y for each fuzzy regular open set λ in X.
(v) fuzzy α-open [8] if $f(\lambda)$ is a fuzzy α-open set of Y for each fuzzy open subset λ of X.
(vi) fuzzy θ-continuous [15] (resp. fuzzy weakly θ-continuous [15]) if for each fuzzy point x_p and each open nbd λ of of $f(x_p)$, there is an fuzzy open nbd μ of x_p such that $f(Cl(\mu)) \leq Cl(\lambda)$ (resp. $f(Int(Cl(\mu))) \leq Cl(\lambda)$).
(vii) fuzzy contra-open (resp. fuzzy contra closed) if $f(\lambda)$ is a fuzzy closed (resp. fuzzy open) set of Y for each fuzzy open (resp. fuzzy closed) set λ in X.

DEFINITIONS 1.4. [10]. A fuzzy point x_p in a fts X is said to be a fuzzy θ-cluster point of a fuzzy set λ if and only if for every fuzzy open q-nbd μ of x_p , $Cl(\mu)$ is q-coincident with λ. The set of all fuzzy θ-cluster points of λ is called the fuzzy θ-closure of λ and is denoted by $Cl_\theta(\lambda)$. A fuzzy set λ is fuzzy θ-closed if and only if $\lambda = Cl_\theta(\lambda)$. The complement of a fuzzy θ-closed set is called of fuzzy θ-open and the θ-interior of λ denoted by $Int_\theta(\lambda)$ is defined as :

$$Int_\theta(\lambda) = \{x_p : \text{for some fuzzy open q-nbd , } \beta \text{ of } x_p, Cl(\beta) \leq \lambda\}.$$

LEMMA 1.5. [2].Let λ be a fuzzy set in a fts X, then:
(1) λ is a fuzzy θ-open if and only if $\lambda = Int_\theta(\lambda)$.
(2) $1 - Int_\theta(\lambda) = Cl_\theta(1 - \lambda)$ and $Int_\theta(1 - \lambda) = 1 - Cl_\theta(\lambda)$.
(3) $Cl_\theta(\lambda)$ (resp. $Int_\theta(\lambda)$) is a fuzzy closed set (resp. fuzzy open set) but not necessarily is a fuzzy θ-closed set (resp. fuzzy θ-open set).

RESULT. 1.6. (i) It is easy to see that $Cl(\lambda) \leq Cl_\theta(\lambda)$ and $Int_\theta(\lambda) \leq Int(\lambda)$ for any fuzzy set λ in a fts X.
(ii) For a fuzzy open (resp. fuzzy closed) set λ in a fts X, $Cl(\lambda) = Cl_\theta(\lambda)$ (resp. $Int_\theta(\lambda) = Int(\lambda)$).

2 Fuzzy Weakly Preopen Functions

Now, we define the generalized form of preopen functions in fuzzy setting.

DEFINITION 2.1. A function $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be fuzzy weakly preopen if $f(\lambda) \leq pInt(f(Cl(\lambda)))$ for each fuzzy open subset λ of X.

Clearly, every fuzzy weakly open function is fuzzy preopen and every fuzzy preopen function is fuzzy weakly preopen (since, $f(\lambda) = pInt(f(\lambda)) \leq pInt(f(Cl(\lambda)))$ for each fuzzy open set λ of X.), but the converse is not generally true, as the next example shows.

EXAMPLE 2.2. Let $X = \{a, b, c\}$,$\tau = \{0, A, B, A \cup B, A \cap B, 1\}$ and $\sigma = \{0, E, H, E \cup H, E \cap H, 1\}$. Where A, B, E and H are defined as follows:

$$A(a) = 0.4 \quad , \quad A(b) = 0.7 \quad , A(c) = 0.2;$$

$$B(a) = 0.3 \quad , \quad B(b) = 0.1 \quad , B(c) = 0.6;$$

$$E(a) = 0.5 \quad , \quad E(b) = 0.8 \quad , E(c) = 0.3;$$

$$H(a) = 0.4 \quad , \quad H(b) = 0.2 \quad , H(c) = 0.7.$$

Consider the identity mapping $f : (X, \tau) \to (X, \sigma)$. Clearly f is fuzzy weakly preopen but not fuzzy preopen.

EXAMPLE 2.3. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$, $\tau = \{0, A, 1\}$, $\sigma = \{0, B, H, 1\}$. Fuzzy sets A, B, and H be defined as :

$$A(a) = 0.5 \quad , \quad A(b) = 0.3 \quad , A(c) = 0.2;$$

$$B(x) = 0.9 \quad , \quad B(y) = 1 \quad , B(z) = 0.7;$$

$$H(x) = 0.2 \quad , \quad H(y) = 0.9 \quad , H(z) = 0.3.$$

Then the mapping $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = z$ and $f(b) = x$, $f(c) = y$ is fuzzy weakly preopen but not fuzzy weakly open.

THEOREM 2.4. For a function $f : (X, \tau_1) \to (Y, \tau_2)$, the following conditions are equivalent:
(i) f is fuzzy weakly preopen,
(ii) $f(Int_\theta(\lambda)) \leq pInt(f(\lambda))$ for every fuzzy subset λ of X,

(iii) $Int_\theta(f^{-1}(\beta)) \leq f^{-1}(pInt(\beta))$ for every fuzzy subset β of Y,
(iv) $f^{-1}(pCl(\beta)) \leq Cl_\theta(f^{-1}(\beta))$ for every fuzzy subset β of Y.

PROOF. $(i) \rightarrow (ii)$: Let λ be any fuzzy subset of X and x_p a fuzzy point in $Int_\theta(\lambda)$. Then , there exists a fuzzy open q-nbd γ *of* x_p such that $\gamma \leq Cl(\gamma) \leq \lambda$. Then, $f(\gamma) \leq f(Cl(\gamma)) \leq f(\lambda)$. Since f is fuzzy weakly preopen, $f(\gamma) \leq pInt(f(Cl(\gamma))) \leq pInt(f(\lambda))$. It implies that $f(x_p)$ is a point in $pInt(f(\lambda))$. This shows that $x_p \in f^{-1}(pInt(f(\lambda)))$. Thus $Int_\theta(\lambda) \leq f^{-1}(pInt(f(\lambda)))$, and so $f(Int_\theta(\lambda)) \leq pInt(f(\lambda))$.

$(ii) \rightarrow (i)$: Let μ be a fuzzy open set in X. As $\mu \leq Int_\theta(Cl(\mu))$ implies , $f(\mu) \leq f(Int_\theta(Cl(\mu))) \leq pInt(f(Cl(\mu)))$. Hence f is fuzzy weakly preopen.

$(ii) \rightarrow (iii)$: Let β be any fuzzy subset of Y. Then by (ii), $f(Int_\theta(f^{-1}(\beta))) \leq pInt(\beta)$. Therefore $Int_\theta(f^{-1}(\beta)) \leq f^{-1}(pInt(\beta))$.

$(iii) \rightarrow (ii)$: This is obvious.

$(iii) \rightarrow (iv)$: Let β be any fuzzy subset of Y. Using (iii), we have

$$1-Cl_\theta(f^{-1}(\beta)) = Int_\theta(1-f^{-1}(\beta)) = Int_\theta(f^{-1}(1-\beta)) \leq f^{-1}(pInt(1-\beta)) =$$

$f^{-1}(1 - pCl(\beta)) = 1 - (f^{-1}(pCl(\beta))$. Therefore, we obtain $f^{-1}(pCl(\beta)) \leq Cl_\theta(f^{-1}(\beta))$.

$(iv) \rightarrow (iii)$: Similary we obtain, $1 - f^{-1}(pInt(\beta)) \leq 1 - Int_\theta(f^{-1}(\beta))$, for every fuzzy subset β of Y, i.e., $Int_\theta(f^{-1}(\beta)) \leq f^{-1}(pInt(\beta))$.

THEOREM 2.5. If X is a fuzzy regular space, then for a function $f : (X, \tau_1) \rightarrow (Y, \tau_2)$, the following conditions are equivalent:
(i) f is fuzzy weakly preopen,
(ii) For each fuzzy θ-open set λ in X, $f(\lambda)$ is fuzzy preopen in Y,
(iii) For any fuzzy set β of Y and any fuzzy θ-closed set λ in X containing $f^{-1}(\beta)$, there exists a fuzzy preclosed set δ in Y containing β such that $f^{-1}(\delta) \leq \lambda$.

PROOF. $(i) \rightarrow (ii)$: Let λ be a fuzzy θ-open set in X. Then $1 - f(\lambda)$ is a fuzzy set in Y and by (i) and Theorem 2.4(iv), $f^{-1}(pCl(1 - f(\lambda))) \leq Cl_\theta(f^{-1}(1 - f(\lambda)))$. Therefore, $1 - f^{-1}(pInt(f(\lambda))) \leq Cl_\theta(1 - \lambda) = 1 - \lambda$. Then, we have $\lambda \leq f^{-1}(pInt(f(\lambda)))$ which implies $f(\lambda) \leq pInt(f(\lambda))$. Hence $f(\lambda)$ is fuzzy preopen in Y.

$(ii) \rightarrow (iii)$: Let β be any fuzzy set in Y and λ be a fuzzy θ-closed set in X such that $f^{-1}(\beta) \leq \lambda$. *Since* $1 - \lambda$ *is fuzzy* θ-open in X, by (ii), $f(1 - \lambda)$

is fuzzy preopen in Y. Let $\delta = 1 - f(1-\lambda)$. Then δ is fuzzy preclosed and $\beta \leq \delta$. Now, $f^{-1}(\delta) = f^{-1}(1 - f(1-\lambda)) = 1 - f^{-1}(f(1-\lambda)) \leq \lambda$.

$(iii) \to (i)$: Let β be any fuzzy set in Y. Then by Corollary 3.6 of [10] $\lambda = Cl_\theta(f^{-1}(\beta))$ is fuzzy θ-closed set in X and $f^{-1}(\beta) \leq \lambda$. Then there exists a fuzzy preclosed set δ in Y containing β such that $f^{-1}(\delta) \leq \lambda$. Since δ is fuzzy preclosed $f^{-1}(pCl(\beta)) \leq f^{-1}(\delta) \leq Cl_\theta(f^{-1}(\beta))$. Therefore by Theorem 2.4, f is a fuzzy weakly preopen function.

THEOREM 2.6. For a function $f : (X, \tau_1) \to (Y, \tau_2)$, the following statement are equivalent:
(i) f is fuzzy weakly preopen,
(ii) For each x_p fuzzy point in X and each fuzzy open subset μ of X containing x_p, there exists a fuzzy preopen set δ containing $f(x_p)$ such that $\delta \leq f(Cl(\mu))$.

PROOF. $(i) \to (ii)$: Let $x_p \in X$ and μ be a fuzzy open set in X containing x_p. Since f is fuzzy weakly preopen $f(\mu) \leq pInt(f(Cl(\mu)))$. Let $\delta = pInt(f(Cl(\mu)))$. Hence $\delta \leq f(Cl(\mu))$, with δ containing $f(x_p)$.

$(ii) \to (i)$: Let μ be a fuzzy open set in X and let $y_p \in f(\mu)$. It following from (ii) that $\delta \leq f(Cl(\mu))$ for some δ fuzzy preopen in Y containing y_p. Hence we have , $y_p \in \delta \leq pInt(f(Cl(\mu)))$. This shows that $f(\mu) \leq pInt(f(Cl(\mu)))$, i.e., f is a fuzzy weakly preopen function.

THEOREM 2.7. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijective function. Then the following statements are equivalent.
(i) f is fuzzy weakly preopen,
(ii) $pCl(f(\lambda)) \leq f(Cl(\lambda))$ for each λ fuzzy open subset of X,
(iii) $pCl(f(Int(\beta)) \leq f(\beta)$ for each β fuzzy closed subset of X.

PROOF. $(i) \to (iii)$: Let β be a fuzzy closed set in X. Then we have $f(1-\beta) = 1 - f(\beta) \leq pInt(f(Cl(1-\beta)))$ and so $1 - f(\beta) \leq 1 - pCl(f(Int(\beta)))$. Hence $pCl(f(Int(\beta))) \leq f(\beta)$.

$(iii) \to (ii)$: Let λ be a fuzzy open set in X. Since $Cl(\lambda)$ is a fuzzy closed set and $\lambda \leq Int(Cl(\lambda))$ by (iii)we have $pCl(f(\lambda)) \leq pCl(f(Int(Cl(\lambda))) \leq f(Cl(\lambda))$.

$(ii) \to (iii)$: Obvious.

$(iii) \to (i)$: Similar to $(iii) \to (ii)$.

The proof of the following theorem is mostly straightforward and hence is omitted.

THEOREM 2.8. For a function $f : (X, \tau_1) \to (Y, \tau_2)$ the following conditions are equivalent:
(i) f is fuzzy weakly preopen.
(ii) $f(Int(\beta)) \leq pInt(f(\beta))$, for each fuzzy closed subset β of X.
(iii) $f(Int(Cl(\lambda))) \leq pInt(f(Cl(\lambda)))$, for each fuzzy open subset λ of X.
(iv) $f(\lambda) \leq pInt(f(Cl(\lambda)))$, for every fuzzy preopen subset λ of X.
(v) $f(\lambda) \leq pInt(f(Cl(\lambda)))$, for every fuzzy α-open subset λ of X.

Now, we give a definition of strong continuity. This definition when combined with fuzzy weak preopenness implies fuzzy preopenness.

DEFINITION 2.9. A function $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be fuzzy strongly continuous, if for every fuzzy subset λ of X, $f(Cl(\lambda)) \leq f(\lambda)$.

LEMMA 2.10. If $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy strongly continuous, then $pInt(f(Cl(\lambda))) \leq f(\lambda)$, but the converse does not hold . Example 2.3 above serves the purpose.

THEOREM 2.11. If $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy weakly preopen and fuzzy strongly continuous, then f is fuzzy preopen.

PROOF. Let λ be a fuzzy open subset of X. Since f is fuzzy weakly preopen $f(\lambda) \leq pInt(f(Cl(\lambda)))$. However, because f is fuzzy strongly continuous, $f(\lambda) \leq pInt(f(\lambda))$ and therefore $f(\lambda)$ is fuzzy preopen.

A function $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be fuzzy contra preclosed if $f(\lambda)$ is a fuzzy preopen set of Y, for each fuzzy closed set λ in X.

THEOREM 2.12. If $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy contra preclosed, then f is a fuzzy weakly preopen function .

PROOF. Let λ be a fuzzy open subset of X. Then, we have $f(\lambda) \leq f(Cl(\lambda))) = pInt(f(Cl(\lambda)))$.

The converse of Theorem 2.12 does not hold.

EXAMPLE. 2.13. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$.
Define fuzzy sets A and B as :

$$A(a) = 0 \quad , \quad A(b) = 0.2 \quad , \quad A(c) = 0.7 \quad ;$$

$$B(x) = 0 \quad , \quad B(y) = 0.2 \quad , \quad B(z) = 0.2 \quad .$$

Let $\tau = \{0, A, 1\}$, $\sigma = \{0, B, 1\}$. Then the mapping $f : (X, \tau) \to (Y, \sigma)$ defined as : $f(a) = x$, $f(b) = y$ and $f(c) = y$ is fuzzy weakly preopen but not fuzzy contra preclosed .

THEOREM 2.14. Let X be a fuzzy regular space. Then $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy weakly preopen if and only if f is fuzzy preopen.

PROOF. The sufficiency is clear. For the necessity, let λ be a non-null fuzzy open subset of X. For each x_p fuzzy point in λ, let μ_{x_p} be an fuzzy open set such that $x_p \in \mu_{x_p} \leq Cl(\mu_{x_p}) \leq \lambda$. Hence we obtain that $\lambda = \cup\{\mu_{x_p} : x_p \in \lambda\} = \cup\{Cl(\mu_{x_p}) : x_p \in \lambda\}$ and , $f(\lambda) = \cup\{f(\mu_{x_p}) : x_p \in \lambda\} \leq \cup\{pInt(f(Cl(\mu_{x_p}))) : x_p \in \lambda\} \leq pInt(f(\cup\{Cl(\mu_{x_p}) : x_p \in \lambda\}) = pInt(f(\lambda))$. Thus f is fuzzy preopen.

THEOREM 2.15. If $f : (X, \tau_1) \to (Y, \tau_2)$ is an f.a.o.N function, then it is a fuzzy weakly preopen function.

PROOF. Let λ be a fuzzy open set in X. Since f is f.a.o.N and $Int(Cl(\lambda))$ is fuzzy regular open, $f(Int(Cl(\lambda)))$ is fuzzy open in Y and hence $f(\lambda) \leq f(Int(Cl(\lambda)) \leq Int(f(Cl(\lambda))) \leq pInt(f(Cl(\lambda)))$. This shows that f is fuzzy weakly preopen.

The converse of Theorem 2.15 is not true in general.Example 2.2 above serves the purpose.

LEMMA 2.16. [9] If $f : (X, \tau_1) \to (Y, \tau_2)$ is a fuzzy continuous function, then for any fuzzy subset λ of X , $f(Cl(\lambda)) \leq Cl(f(\lambda))$

THEOREM 2.17. If $f : (X, \tau_1) \to (Y, \tau_2)$ is a fuzzy weakly preopen and fuzzy continuous function, then f is a fuzzy α-open function.

PROOF. Let λ be a fuzzy open set in X. Then by fuzzy weak preopenness of f, $f(\lambda) \leq pInt(f(Cl(\lambda)))$. Since f is fuzzy continuous $f(Cl(\lambda)) \leq Cl(f(\lambda))$. Hence we obtain that, $f(\lambda) \leq pInt(f(Cl(\lambda))) \leq pInt(Cl(f(\lambda))) \leq$

$Int(Cl(Int(f(\lambda))))$. Therefore, $f(\lambda) \leq Int(Cl(Int(f(\lambda)))$ which shows that $f(\lambda)$ is a fuzzy α-open set in Y. Thus by definition 1.3, f is a fuzzy α-open function.

Since, every fuzzy strongly continuous function is fuzzy continuous we have the following corollary,

COROLLARY 2.18. If $f : (X, \tau_1) \to (Y, \tau_2)$ is a fuzzy weakly preopen and fuzzy strongly continuous function. Then f is a fuzzy α-open function.

DEFINITION 2.19. Two non-empty fuzzy sets λ and β in a fuzzy topological spaces X (i.e., neither λ nor β is 0_X) are said to be fuzzy pre-separated if $\lambda \bar{q} pCl(\beta)$ and $\beta \bar{q} pCl(\lambda)$ or equivalently if there exist two fuzzy preopen sets μ and ν such that $\lambda \leq \mu$, $\beta \leq \nu$, $\lambda \bar{q} \nu$ and $\beta \bar{q} \mu$.

A fuzzy topological space X which cannot be expressed as the union of two fuzzy pre-separated sets is said to be a fuzzy pre-connected space.

THEOREM 2.20. If $f : (X, \tau_1) \to (Y, \tau_2)$ is an injective fuzzy weakly preopen function of a space X onto a fuzzy pre-connected space Y, then X is fuzzy connected.

PROOF. If possible, let X be not connected. Then there exist fuzzy separated sets β and γ in X such that $X = \beta \cup \gamma$. Since β and γ are fuzzy separated, there exist two fuzzy open sets μ and ν such that $\beta \leq \mu$, $\gamma \leq \nu$, $\beta \bar{q} \nu$ and $\gamma \bar{q} \mu$ .Hence we have $f(\beta) \leq f(\mu)$, $f(\gamma) \leq f(\nu)$, $f(\beta) \bar{q} f(\nu)$ and $f(\gamma) \bar{q} f(\mu)$. Since f is fuzzy weakly preopen, we have $f(\mu) \leq pInt(f(Cl(\mu)))$ and $f(\nu) \leq pInt(f(Cl(\nu)))$ and since μ and ν are fuzzy open and also fuzzy closed, we have $f(Cl(\mu)) = f(\mu)$, $f(Cl(\nu)) = f(\nu)$. Hence $f(\mu)$ and $f(\nu)$ are fuzzy preopen in Y. Therefore , $f(\beta)$ and $f(\gamma)$ are fuzzy pre-separated sets in Y and $Y = f(X) = f(\beta \cup \gamma) = f(\beta) \cup f(\gamma)$. Hence this contrary to the fact that Y is fuzzy pre-connected. Thus X is fuzzy connected.

DEFINITION 2.21. A space X is said to be fuzzy hyper-connected if every non-null fuzzy open subset of X is fuzzy dense in X.

THEOREM 2.22. If X is a fuzzy hyper-connected space, then a function $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy weakly preopen if and only if $f(X)$ is fuzzy preopen in Y.

PROOF. The sufficiency is clear. For the necessity observe that for any fuzzy open subset λ of $X, f(\lambda) \leq f(X) = pInt(f(X) = pInt(f(Cl(\lambda)))$.

3 Fuzzy Weakly Preclosed Functions

Now, we define the generalized form of preclosed functions in fuzzy setting.

DEFINITION 3.1. A function $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be fuzzy weakly preclosed if $pCl(f(Int(\beta))) \leq f(\beta)$ for each fuzzy closed subset β of X.

Clearly, every fuzzy preclosed function is fuzzy weakly preclosed function since $pCl(f(Int(\beta))) \leq pCl(f(\beta)) = f(\beta)$ for every fuzzy closed subset β of X, but the converse is not generally true, as the next example shows.

EXAMPLE 3.2. Let $X = \{a, b\}$ and $Y = \{x, y\}$. Fuzzy sets A and B are defined as :

$$A(x) = 0.4, \quad A(y) = 0.3;$$

$$B(a) = 0.5, \quad B(b) = 0.6.$$

Let $\tau_1 = \{0, B, 1_X\}$ and $\tau_2 = \{0, A, 1_Y\}$. Then the function $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ defined by $f(a) = x$, $f(b) = y$ is fuzzy weakly preclosed but not fuzzy preclosed.

Recall that, every fuzzy closed function is fuzzy α-closed and every fuzzy α-closed function is fuzzy preclosed, but the reverse implications not be true in general [5].

From result above and Example 3.2, we have the following diagrama and the converses of these implication do not hold, in general as is showd.

$$\begin{array}{ccc} \text{fuzzy closed function} & \longrightarrow & \text{fuzzy } \alpha\text{-closed function} \\ \downarrow & & \downarrow \\ \text{fuzzy weakly preclosed function} & \longleftarrow & \text{fuzzy preclosed function} \end{array}$$

THEOREM 3.3. For a function $f : (X, \tau_1) \to (Y, \tau_2)$. The following conditions are equivalent.
(i) f is fuzzy weakly preclosed,
(ii) $pCl(f(\lambda)) \leq f(Cl(\lambda))$ for every fuzzy open set λ in X .

PROOF. $(i) \to (ii)$: Let λ be any fuzzy open subset of X. Then, $pCl(f(\lambda)) = pCl(f(Int(\lambda))) \leq pCl(f(Int(Cl(\lambda))) \leq f(Cl(\lambda))$.

$(ii) \to (i)$: Let β be any fuzzy closed subset of X. Then, $pCl(f(Int(\beta))) \leq f(Cl(Int(\beta))) \leq f(Cl(\beta)) = f(\beta)$.

The proof of the following theorem, is mostly straightforward and hence is omitted.

THEOREM 3.4. For a function $f : (X, \tau_1) \to (Y, \tau_2)$ the following conditions are equivalent:
(i) f is fuzzy weakly preclosed.
(ii) $pCl(f(\beta)) \leq f(Cl(\beta))$ for each fuzzy open subset β of X.
(iii) $pCl(f(Int(\beta))) \leq f(\beta)$ for each fuzzy closed subset β of X.
(iv) $pCl(f(Int(\beta))) \leq f(\beta)$ for each fuzzy preclosed subset β of X.
(v) $pCl(f(Int(\beta))) \leq f(\beta)$ for every fuzzy α-closed subset β of X.

THEOREM 3.5. For a function $f : (X, \tau_1) \to (Y, \tau_2)$ the following conditions are equivalent:
(i) f is fuzzy weakly preclosed.
(ii) $pCl(f(\lambda)) \leq f(Cl(\lambda))$ for each fuzzy regular open subset λ of X.
(iii) For each fuzzy subset β of Y and each fuzzy open set μ in X with $f^{-1}(\beta) \leq \mu$, there exists a fuzzy preopen set δ in Y with $\beta \leq \delta$ and $f^{-1}(\delta) \leq Cl(\mu)$.
(iv) For each fuzzy point y_p in Y and each fuzzy open set μ in X with $f^{-1}(y_p) \leq \mu$, there exists a fuzzy preopen set δ in Y containing y_p and $f^{-1}(\delta) \leq Cl(\mu)$.
(v) $pCl(f(Int(Cl(\lambda)))) \leq f(Cl(\lambda))$ for each fuzzy set λ in X.
(vi) $pCl(f(Int(Cl_\theta(\lambda)))) \leq f(Cl_\theta(\lambda))$ for each fuzzy set λ in X.
(vii) $pCl(f(\lambda)) \leq f(Cl(\lambda))$ for each fuzzy preopen set λ in X.

PROOF. It is clear that: $(i) \to (vi), (iii) \to (iv), and (i) \to (v) \to (vii) \to (ii) \to (i)$. To show that $(ii) \to (iii)$: Let β be a fuzzy subset of Y and let μ be fuzzy open in X with $f^{-1}(\beta) \leq \mu$. Then $f^{-1}(\beta)\bar{q}Cl(1_X - Cl(\mu))$ and consequently, $\beta\bar{q}f(Cl(1_X - Cl(\mu)))$. Since $1_X - Cl(\mu)$ is fuzzy regular open, $\beta\bar{q}pCl(f(1_X - Cl(\mu)))$ by (ii). Let $\delta = 1_Y - pCl(f(1_X - Cl(\mu)))$. Then δ is fuzzy preopen with $\beta \leq \delta$ and $f^{-1}(\delta) \leq 1_X - f^{-1}(pCl(f(1_X - Cl(\mu)))) \leq 1_X - f^{-1}f(1_X - Cl(\mu)) \leq Cl(\mu)$.

$(vi) \rightarrow (i)$: It is suffices see that $Cl_\theta(\lambda) = Cl(\lambda)$ for every fuzzy open sets λ in X.

$(iv) \rightarrow (i)$: Let β be fuzzy closed in $X$and let $y_p \in 1_Y - f(\beta)$. Since $f^{-1}(y_p) \leq 1_X - \beta$, there exists a fuzzy preopen δ in $Y$with $y_p \in \delta$ and $f^{-1}(\delta) \leq Cl(1_X - \beta) = 1_X - Int(\beta)$ by (iv). Therefore $\delta\bar{q}f(Int(\beta))$, so that $y_p \in 1_Y - pCl(f(Int(\beta)))$. Thus $(iv) \rightarrow (i)$. Finally, for

$(vi) \rightarrow (vii)$: Note that $Cl_\theta(\lambda) = Cl(\lambda)$ for each fuzzy preopen subset λ in X [6].

THEOREM 3.6. If $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is fuzzy weakly preclosed, then for each fuzzy point y_p in Y and each fuzzy open q-nbd μ of $f^{-1}(y_p)$ in X, there exists a fuzzy preopen q-nbd δ of y_p in Y, such that $f^{-1}(\delta) \leq Cl(\mu)$.

PROOF. Let μ be any fuzzy open q-nbd of $f^{-1}(y_p)$ in X. Then $\mu(x)+p > 1$ and hence there exists a positive real number α such that $\mu(x) > \alpha > 1-p$. Then μ is a fuzzy open q-nbd of $f^{-1}(y_\alpha)$. By Theorem 3.5(iv) there exists a fuzzy preopen set δ containing y_α in Y such that $f^{-1}(\delta) \leq Cl(\mu)$. Now, $\delta(y) > \alpha$ and hence $\delta(y) > 1-p$. Thus δ is a fuzzy preopen q-nbd of y_p.

DEFINITION 3.7.A fuzzy set λ in a fts (X, τ) is called pre-q-nbd of x_α if there exist a fuzzy preopen subset μ in X such that $x_\alpha q \mu \leq \lambda$

THEOREM 3.8. In a fts (X, τ) a fuzzy point $x_\alpha \in pCl(\lambda)$ if and only if every pre-q-nbd of x_α is quasi-coincident with λ.

PROOF.Suppose $x_\alpha \in pCl(\lambda)$ and if possible, let there exist a pre-q-nbd μ of x_α such that $\mu\bar{q}\lambda$. Then there exist a fuzzy preopen set ν in X such that $x_\alpha\, q\, \nu \leq \mu$ which shows that $\nu\bar{q}\lambda$ and hence $\lambda \leq 1-\nu$. As $1-\nu$ is fuzzy preclosed, $pCl(\lambda) \leq 1-\nu$. Since $x_\alpha \notin 1-\nu$, we obtain $x_\alpha \notin pCl(\lambda)$ which is a contradiction.

Conversely, suppose $x_\alpha \notin pCl(\lambda)$. Then there exists a fuzzy preclosed set $\lambda \leq \beta$ such that $x_\alpha \notin \beta$. We then have $x_\alpha\, q(1-\beta) \in FPO(X, \tau)$ and $\lambda\bar{q}(1-\beta)$.

Next we investigate conditions under which fuzzy weakly preclosed functions are fuzzy preclosed.

THEOREM 3.9. If $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy weakly preclosed and if for each fuzzy closed subset β of X and each fiber $f^{-1}(y_p) \leq 1_X - \beta$ there exists a fuzzy open q-nbd μ of X such that $f^{-1}(y_p) \leq \mu \leq Cl(\mu) \leq 1_X - \beta$. Then f is fuzzy preclosed.

PROOF.Let β is any fuzzy closed subset of X and let $y_p \in 1_Y - f(\beta)$. Then $f^{-1}(y_p)\bar{q}\beta$ and hence $f^{-1}(y_p) \leq 1_X - \beta$. By hypothesis, there exists a fuzzy open q-nbd μ of X such that $f^{-1}(y_p) \leq \mu \leq Cl(\mu) \leq 1_X - \beta$. Since f is fuzzy weakly preclosed by Theorem 3.6, there exists a fuzzy preopen q-nbd ν in Y with $y_p \in \nu$ and $f^{-1}(\nu) \leq Cl(\mu)$. Therefore , we obtain $f^{-1}(\nu)\bar{q}\beta$ and hence $\nu\bar{q}f(\beta)$, this shows that $y_p \notin pCl(f(\beta))$. Therefore, $f(\beta)$ is fuzzy preclosed in Y and f is fuzzy preclosed .

THEOREM 3.10. (i) If $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy preclosed and fuzzy contra-closed, then f is fuzzy weakly preclosed.
(ii) If $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy contra-open , then f is fuzzy weakly preclosed.

PROOF. (i) Let β be a fuzzy closed subset of X. Since f is fuzzy preclosed $Cl(Int(f(\beta))) \leq f(\beta)$ and since f is fuzzy contra-closed $f(\beta)$ is fuzzy open. Therefore $pCl(f(Int(\beta))) \leq pCl(f(\beta)) \leq Cl(Int(f(\beta))) \leq f(\beta)$.

(ii) Let β be a fuzzy closed subset of X. Then,
$pCl(f(Int(\beta))) \leq f(Int(\beta)) \leq f(\beta)$.

THEOREM 3.11. If $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy weakly preclosed , then for every fuzzy subset β in Y and every fuzzy open set λ in X with $f^{-1}(\beta) \leq \lambda$, there exists a fuzzy preclosed set δ in Y such that $\beta \leq \delta$ and $f^{-1}(\delta) \leq Cl(\lambda)$.

PROOF. Let β be a fuzzy subset of Y and let λ be a fuzzy open subset of X with $f^{-1}(\beta) \leq \lambda$. Put $\delta = pCl(f(Int(Cl(\lambda)))$, then δ is a fuzzy preclosed set of Y such that $\beta \leq \delta$ since $\beta \leq f(\lambda) \leq f(Int(Cl(\lambda)) \leq pCl(f(Int(Cl(\lambda))) = \delta$. And since f is fuzzy weakly preclosed, $f^{-1}(\delta) \leq Cl(\lambda)$ (Theorem 3.5).

COROLLARY 3.12. If $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy weakly preclosed , then for every fuzzy point y_p in Y and every fuzzy open set λ in X with

$f^{-1}(y_p) \leq \lambda$, there exists a fuzzy preclosed set δ in Y containing y_p such that $f^{-1}(\delta) \leq Cl(\lambda)$.

A fuzzy set β in a fts X is fuzzy θ-compact if for each cover Ω of β by fuzzy open q-nbd μ in X, there is a finite family $\mu_1, ..., \mu_n$ in Ω such that $\beta \leq Int(\cup\{Cl(\mu_i) : i = 1, 2, ..., n\})$.

THEOREM 3.13. If $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy weakly preclosed with all fibers fuzzy θ-closed, then $f(\beta)$ is fuzzy preclosed for each fuzzy θ-compact β in$X$.

PROOF. Let β be fuzzy θ-compact and let $y_p \in 1_Y - f(\beta)$. Then $f^{-1}(y_p)\bar{q}\beta$ and for each $x_p \in \beta$ there is a fuzzy open q-nbd μ_{x_p} containing x_p in X and $Cl(\mu_{x_p})\bar{q}f^{-1}(y_p)$. Clearly $\Omega = \{\mu_{x_p} : x_p \in \beta\}$ is a fuzzy open q-nbd cover of β and since β is fuzzy θ-compact, there is a finite family $\{\mu_{x_1}, ..., \mu_{x_n}\}$ in Ω such that $\beta \leq Int(\lambda)$, where $\lambda = \cup\{Cl(\mu_{x_i}) : i = 1, ..., n\}$. Since f is fuzzy weakly preclosed by Theorem 3.6 there exists a fuzzy preopen q-nbd δ in Y with $f^{-1}(y_p) \leq f^{-1}(\delta) \leq Cl(1_X - \lambda) = 1_X - Int(\lambda) \leq 1_X - \beta$. Therefore $y_p \in \delta$ and $\delta\bar{q}f(\beta)$. Thus $y_p \in 1_Y - pCl(f(\beta))$. This shows that $f(\beta)$ is fuzzy preclosed.

Two non empty fuzzy subsets λ and β in X are strongly fuzzy separated if there exist fuzzy open sets μ and ν in X with $\lambda \leq \mu$ and $\beta \leq \nu$ and $Cl(\mu)\bar{q}Cl(\nu)$. If λ and β are fuzzy singleton sets we may speak of fuzzy points being strongly fuzzy separated . We will use the fact that in a fuzzy normal space, disjoint fuzzy closed sets are strongly fuzzy separated.

Recall that space X is said to be fuzzy pre-Hausdorff or in short fuzzy pre-T_2 [17] if for every pair of fuzzy points x_p and x_q with different supports, there exist two fuzzy preopen sets λ and β such that $x_p \leq \lambda \leq x_q^c$ and $x_q \leq \beta \leq x_p^c$ and $\lambda\bar{q}\beta$.

THEOREM 3.14.If $f : (X, \tau_1) \to (Y, \tau_2)$ is a fuzzy weakly preclosed surjection and all pairs of disjoint fibers are strongly fuzzy separated, then Y is fuzzy pre-T_2.

PROOF.Let y_p and y_q be two fuzzy points in Y. Let μ and ν be fuzzy open sets in X such that $f^{-1}(y_p) \leq \mu$ and $f^{-1}(y_q) \leq \nu$ respectively with

$Cl(\mu)\bar{q}Cl(\nu)$. By fuzzy weak preclosedness (Theorem 2.5(iv)) there are fuzzy preopen sets λ and β in Y such that $y_p \leq \lambda$ and $y_q \leq \beta$, $f^{-1}(\lambda) \leq Cl(\mu)$ and $f^{-1}(\beta) \leq Cl(\nu)$. Therefore $\lambda\bar{q}\beta$, because $Cl(\mu)\bar{q}Cl(\nu)$ and f surjective. Then Y is fuzzy pre-T_2.

COROLLARY 3.15. If $f : (X, \tau_1) \to (Y, \tau_2)$ is a weakly fuzzy preclosed surjection with all fuzzy fibers closed and X is fuzzy normal, then Y is a fuzzy $pre - T_2$ space.

DEFINITION 3.16. A family $\{\lambda_\alpha : \alpha \in \Omega\}$ of fuzzy open subsets (resp. fuzzy preclosed subsets) of a fuzzy topological space (X, τ) is a fuzzy open cover (resp. fuzzy preclosed cover) if $\cup\{\lambda_\alpha : \alpha \in \Omega\} = X$. An fts X is said to be fuzzy almost compact [7,5] (resp. fuzzy p-compact) if every fuzzy open cover (resp. fuzzy preclosed cover) contains a finite subfamily $\{\lambda_{\alpha_i} : i = 1, 2, ..., n\}$ such that $X = \overset{n}{\underset{i}{\cup}} Cl(\lambda_{\alpha_i})$. A fuzzy subset λ of a fts X is fuzzy almost compact relative to X (resp. fuzzy p-compact relative to X) if every cover of λ by fuzzy open (resp. fuzzy preclosed) sets of X has a finite subfamily whose fuzzy closures cover X.

Recall that a fts (X, τ) is said to be fuzzy extremally disconnected if the closure of every fuzzy open set of X is fuzzy open in X [6].

A fts (X, τ) satisfies the property (po) if the finite intersection of preopen sets is preopen.

LEMMA 3.17. A function $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy open if and only if for each fuzzy subset β of Y , $f^{-1}(Cl(\beta)) \leq Cl(f^{-1}(\beta))$ [18].

THEOREM 3.18. Let X be an fuzzy extremally disconnected space that satisfies the property (po). Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a fuzzy open and fuzzy weakly preclosed function which is one-one and such that $f^{-1}(y_p)$ is fuzzy almost compact relative to X for each fuzzy point y_p in Y. If λ is fuzzy p-compact relative to Y. Then $f^{-1}(\lambda)$ is fuzzy almost compact.

PROOF. Let $\{\nu_\beta : \beta \in I\}$, I being the index set be a fuzzy open cover of $f^{-1}(\lambda)$. Then for each $y_p \in \lambda \cap f(X)$, $f^{-1}(y_p) \leq \cup\{Cl(\nu_\beta) : \beta \in I(y_p)\} = \delta_{y_p}$ for some finite subfamily $I(y_p)$ of I. Since X is fuzzy extremally disconnected each $Cl(\nu_\beta)$ is fuzzy open, hence δ_{y_p} is fuzzy open in X. So by Corollary 3.12, there exists a fuzzy preclosed set μ_{y_p} containing y_p such that $f^{-1}(\mu_{y_p}) \leq Cl(\delta_{y_p})$. Then, $\{\mu_{y_p} : y_p \in \lambda \cap f(X)\} \cup \{1_Y - f(X)\}$ is a fuzzy preclosed cover of λ, $\lambda \leq \cup\{Cl(\mu_{y_p}) : y_p \in K\} \cup \{Cl(1_Y - f(X))\}$ for some finite fuzzy subset K of $\lambda \cap f(X)$. Hence and by Lemma 3.17, $f^{-1}(\lambda) \leq \cup\{f^{-1}(Cl(\mu_{y_p}) : y_p \in K\} \cup \{f^{-1}(Cl(1_Y - f(X))\} \leq \cup\{Cl(f^{-1}(\mu_{y_p}) : y_p \in K\} \cup \{Cl(f^{-1}(1_Y - f(X)))\} \leq \{Cl(f^{-1}(\mu_{y_p})) : y_p \in K\}$, so $f^{-1}(\lambda) \leq \cup\{Cl(\nu_\beta) : \beta \in I(y_p), y_p \in K\}$. Therefore $f^{-1}(\lambda)$ is fuzzy almost compact.

COROLLARY 3.19. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be as in Theorem 3.18. If Y is fuzzy p-compact, then X is fuzzy almost compact.

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