# The functional representation of $\alpha$, $\varepsilon$, $\sigma$, $\theta$ operators by characteristic equations

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This paper deals with the problem of the functional representation of $\alpha$, $\varepsilon$, $\sigma$, and $\theta$ operators. We solve this problem by finding solutions to the characteristic equations based on the triangular norm and conorm.

**Keywords**: t-norms, t-conorms, characteristic equations, fuzzy implications.

**Introduction**. E. Sanchez [1,2] proposed a method for solving fuzzy relation equations. Many applications can be derived using this method. The solution was based on $\alpha$, $\varepsilon$, and $\sigma$ - operators.

We recall that $\alpha$ stands for $\alpha$ - operation defined by

$$x\alpha\ y = \begin{cases} 1, x \le y \\ y, x > y \end{cases}.$$

The definitions $\varepsilon$ and $\sigma$ - operators are given via the following expressions:

$$x\varepsilon\ y = \begin{cases} 0, x \ge y \\ y, x < y \end{cases} \text{ and } x\sigma\ y = \begin{cases} y, x \ge y \\ 0, x < y \end{cases}.$$

Let $f$ and $g$ be additive generators of Archimedian $t$ – norm $T_A$ and Archimedian $t$ – conorm $\perp_A$, then:

$$T_A(x,y) = f^{-1}(\min(f(0), f(x) + f(y))),$$
$$\perp_A(x,y) = g^{-1}(\min(g(1), g(x) + g(y))),$$

where $f:[0;1] \to [0;\infty]$ is continuous and strictly decreasing function, $f(1)=0$; $g:[0;1] \to [0;\infty]$ is continuous and strictly increasing function, $g(0)=0$; $f^{-1}$ and $g^{-1}$ are the inverses of $f$ and $g$.

In this case we denote $\alpha$ and $\varepsilon$ - operators as $\alpha_A$, $\varepsilon_A$ and define [3,4]:

$$x\alpha_A\ y = f^{-1}(f(y) - f(x \vee y));$$
$$x\varepsilon_A\ y = g^{-1}(g(y) - g(x \wedge y)).$$

Obviously $\alpha_A$ and $\varepsilon_A$ are $n$ – dual operators in the sense of De Morgan's law:

$$x\varepsilon_A\ y = n\{n(x)\ \alpha\ n(y)\},$$

where $n(s)$ is a strong negation function.

**Example 1**. Several examples of $t$ – norms, $t$ – conorms and corresponding $\alpha$, $\varepsilon$ - operators are presented below:

| $(i)$ | $(ii)$ | $(iii)$ |
|---|---|---|
| $T(x,y) = x \cdot y$; | $T(x,y) = 0 \vee (x + y - 1)$; | $T(x,y) = \dfrac{x \cdot y}{x + y - x \cdot y}$; |
| $x\alpha\ y = \begin{cases} 1, x \le y \\ \dfrac{y}{x}, x > y \end{cases}$; | $x\alpha\ y = \begin{cases} 1, x \le y \\ 1 - x + y, x > y \end{cases}$; | $x\alpha\ y = \begin{cases} 1, x \le y \\ \dfrac{x \cdot y}{x - y + x \cdot y}, x > y \end{cases}$; |

$\perp(x,y)=x+y-x\cdot y$; $\perp(x,y)=1\wedge(x+y)$; $\perp(x,y)=\dfrac{x+y-2xy}{1-xy}$;

$x\varepsilon\ y=\begin{cases}0, x\geq y\\ \dfrac{y-x}{1-x}, x<y\end{cases}$; $x\varepsilon\ y=\begin{cases}0, x\geq y\\ y-x, x<y\end{cases}$; $x\varepsilon\ y=\begin{cases}0, x\geq y\\ \dfrac{y-x}{1-2x+xy}, x<y\end{cases}$.

We now introduce the $\theta$ - operator

$$x\theta\ y=\begin{cases}y, x\leq y\\ 1, x>y\end{cases}$$

and list below some properties of $\alpha$, $\varepsilon$, $\sigma$, and $\theta$ operators:

1. $x\varepsilon\ y\leq x\alpha\ y$; $x\varepsilon\ y\leq x\vee y$; $x\alpha\ y\geq x\wedge y$
2. $x\sigma\ y\leq x\wedge y\leq x\vee y\leq x\theta\ y$;
3. $\varepsilon$, $\alpha$ are $N-$dual operators in the sense of De Morgan's law $x\varepsilon\ y=N\{N(x)\ \alpha\ N(y)\}$ with the negation defined as $N(s)=1-s$;
4. $\sigma$, $\theta$ are $N-$dual operators in the sense of De Morgan's law $x\sigma\ y=N\{N(x)\ \theta\ N(y)\}$ with the negation defined as $N(s)=1-s$.

In the next section we will represent those operators via characteristic equations by using additive generators $f$ and $g$.

**Results.**

**1. Characteristic equations.**

In this section we introduce the following characteristic equations:

$$T(a\oplus b,z)=b;\quad \perp(a\otimes b,z)=b \quad (1)$$

where $T,\perp$ are a $t-$norm and a $t-$conorm; $\oplus$, $\otimes$ are some operations; $a,b$ are given and $z$ is unknown.

Let us now present a set of propositions that can be used to find a solution of the fuzzy relation equations as well as to derive further applications.

In particular, in the characteristic equations (1) we set $T=T_M=\min$; $\oplus=\vee$ and $\perp=\perp_M=\max$; $\otimes=\wedge$.

Let us denote $z_{\max}=\max\limits_{z\in\Omega} z$; $z_{\min}=\min\limits_{z\in\Omega} z$; $\Omega\subseteq[0;1]$.

The following state results.

**Proposition 1** ($\alpha$ - operation). The solution of the characteristic equation

$$T_M(x\vee y,z)=y \text{ is } z=\begin{cases}(y;1], x\leq y\\ y, x>y\end{cases}$$

From Proposition 1 we have the particular solution $z_{\max}=x\alpha\ y$.

**Proposition 2** ($\varepsilon$ - operation). The solution of the characteristic equation

$$\perp_M(x\wedge y,z)=y \text{ is } z=\begin{cases}[0;y), x\geq y\\ y, x<y\end{cases}$$

From Proposition 2 we have the particular solution $z_{\min}=x\varepsilon\ y$.

In the characteristic equations (1) we set $\oplus = \alpha$ and $\otimes = \varepsilon$. Now, $\theta$, $\sigma$ - operators can be determined by characteristic equations from propositions 3,4.

**Proposition 3** ($\theta$ - operation). The solution of the characteristic equation

$$T_M(x\alpha\ y, z) = y \text{ is } z = \begin{cases} y, x \le y \\ (y;1], x > y \end{cases}$$

From this we have the particular solution $z_{\max} = x\theta\ y$

**Proposition 4** ($\sigma$ - operation). The solution of the characteristic equation

$$\perp_M (x\varepsilon\ y, z) = y \text{ is } z = \begin{cases} y, x \ge y \\ [0; y), x < y \end{cases}$$

From Proposition 4 we have the particular solution $z_{\min} = x\sigma\ y$.

Let us assume that $T, \perp$ are Archimedian triangular norm and conorm. Then in the characteristic equations (1) we set $T = T_A$; $\oplus = \vee$ and $\perp = \perp_A$; $\otimes = \wedge$. In this case the functional representation of $\alpha_A$ and $\varepsilon_A$ can be derived from propositions 5 and 6.

**Proposition 5** ($\alpha_A$ - operation). The solution of the characteristic equation

$$T_A(x \vee y, z) = y \text{ is } z_{\max} = x\alpha_A\ y; \quad x\alpha_A\ y = f^{-1}(f(y) - f(x \vee y))$$

Proof. We consider two cases. Let $x \le y$, then the solution of the equation $T_A(y, z) = y$ is $z_{\max} = 1$. If $x > y$, then by the definition of $T_A$ we have $z = f^{-1}(f(y) - f(x))$.

**Proposition 6** ($\varepsilon_A$ - operation). The solution of the characteristic equation

$$\perp_A (x \wedge y, z) = y \text{ is } z_{\min} = x\varepsilon_A\ y; \quad x\varepsilon_A\ y = g^{-1}(g(y) - g(x \wedge y))$$

Proof. Here we also need to consider two cases. If $x \ge y$, then the solution of the equation $\perp_A (y, z) = y$ is $z_{\min} = 0$. Otherwise, if $x < y$, then according to the definition of $\perp_A$, the solution of this equation is $z = g^{-1}(g(y) - g(x))$.

In order to obtain functional representations of the new operators $\theta_A$ and $\sigma_A$, we introduce the following characteristic equations:

$$T(a \oplus b, z) = a \otimes b; \ \perp (a \otimes b, z) = a \oplus b \quad (2)$$

Returning to $\alpha$, $\varepsilon$ - operators, we consider the pair of $N$ – dual operators $\alpha_G$, $\varepsilon_G$:

$$x\alpha_G\ y = \begin{cases} 1, x \le y \\ 0, x > y \end{cases}; \qquad x\varepsilon_G\ y = \begin{cases} 0, x \ge y \\ 1, x < y \end{cases}; \qquad x\varepsilon_G\ y = N\{N(x)\ \alpha_G\ N(y)\}$$

**Proposition 7** ($\sigma_A$ - operation). The solution of the characteristic equation

$$\perp_M (x\varepsilon_G y, z) = x\alpha_A\ y \text{ is } z_{\min} = x\tilde{\sigma}_A\ y$$

where $x\tilde{\sigma}_A\ y = \begin{cases} f^{-1}(f(y) - f(x)), x \ge y \\ 0, x < y \end{cases}$.

Proof. We first set $x < y$, in this case $z_{\min} = 0$. If $x = y$ then the solution is $z = 1 = f^{-1}(f(x) - f(x))$. Now, when $x > y$, we can derive the solution $z = f^{-1}(f(y) - f(x))$ from the characteristic equation $\max(0, z) = f^{-1}(f(y) - f(x))$. Hence $z_{\min} = x\tilde{\sigma}_A\ y$.

In order to obtain minimal solution of the fuzzy relation equation [2] we need the correction $\tilde{\sigma}_A$ - operator. We add the following condition: $x\tilde{\sigma}_A\, y = 0$ at $y = 0$. As a result we get $\sigma_A$ - operator

$$x\sigma_A\, y = \begin{cases} f^{-1}(f(y) - f(x)), x \geq y > 0 \\ 0, x < y \quad or \quad y = 0 \end{cases}$$

**Proposition 8** ($\theta_A$ - operation). The solution of the characteristic equation

$$T_M(x\alpha_G y, z) = x\varepsilon_A\, y \text{ is } z_{\max} = x\tilde{\theta}_A\, y$$

where $x\tilde{\theta}_A\, y = \begin{cases} g^{-1}(g(y) - g(x)), x \leq y \\ 1, x > y \end{cases}$.

Proof. Suppose that $x > y$, then $z_{\max} = 1$. We can easily see that if $x = y$ then the solution is $z = 0 = g^{-1}(g(x) - g(x))$. Finally, if $x < y$ then the solution $z = g^{-1}(g(y) - g(x))$ can be derived from the characteristic equation $\min(1, z) = g^{-1}(g(y) - g(x))$. Hence $z_{\max} = x\tilde{\theta}_A\, y$.

We now add the following condition: $x\tilde{\theta}_A\, y = 1$ at $y = 1$ and get $\theta_A$ - operator

$$x\theta_A\, y = \begin{cases} g^{-1}(g(y) - g(x)), x \leq y < 1 \\ 1, x > y \quad or \quad y = 1 \end{cases}$$

**Example 2.** Several examples of $t$ – norms, $t$ – conorms and corresponding $\sigma_A, \theta_A$ - operations are presented below:

(*i*)

$T(x,y) = x \cdot y$;

$$x\sigma_A\, y = \begin{cases} 0, x < y \\ \dfrac{y}{x}, x \geq y \end{cases};$$

$\perp(x,y) = x + y - x \cdot y$;

$$x\theta_A\, y = \begin{cases} 1, x > y \\ \dfrac{y-x}{1-x}, x \leq y \end{cases};$$

(*ii*)

$T(x,y) = 0 \vee (x + y - 1)$;

$$x\sigma_A\, y = \begin{cases} 0, x < y \quad or \quad y = 0 \\ 1 - x + y, x \geq y > 0 \end{cases};$$

$\perp(x,y) = 1 \wedge (x + y)$;

$$x\theta_A\, y = \begin{cases} 1, x > y \quad or \quad y = 1 \\ y - x, x \leq y < 1 \end{cases};$$

(*iii*)

$T(x,y) = \dfrac{x \cdot y}{x + y - x \cdot y}$;

$$x\sigma_A\, y = \begin{cases} 0, x < y \\ \dfrac{x \cdot y}{x - y + x \cdot y}, x \geq y \end{cases};$$

$\perp(x,y) = \dfrac{x + y - 2xy}{1 - xy}$;

$$x\theta_A\, y = \begin{cases} 1, x > y \\ \dfrac{y-x}{1-2x+xy}, x \leq y \end{cases}.$$

## 2. Fuzzy implications.

In this section we use the definition of the fuzzy implication presented by [5].

Any function $I : [0;1]^2 \to [0;1]$ is called fuzzy implication if it fulfils the following conditions:

1. $x \leq s \Rightarrow I(x,y) \geq I(s,y), \ \forall x,y,s \in [0;1]$;
2. $y \leq s \Rightarrow I(x,y) \leq I(x,s), \ \forall x,y,s \in [0;1]$;
3. $I(0,y) = 1, \ \forall y \in [0;1]$; $I(x,1) = 1, \ \forall x \in [0;1]$; $I(1,0) = 0$.

For example, it is easy to check that for the function of two variables

$$I_f(x,y) = x\alpha_A\, y = f^{-1}(f(y) - f(x \vee y))$$

the above conditions hold.

Consequently, the $\alpha_A$ - operator is equivalent to an implication operation $I_f$.

**Proposition 9**. If $f(n(s)) = g(s)$, then the function of two variables

$$I_{fg}(x,y) = f^{-1}(g(x) - g(x \wedge y))$$

is the fuzzy implication.

Proof. Since function $I_f(n(y), n(x))$ satisfies above stated conditions, then taking into consideration theorem's condition, we can write

$$I_f(n(y), n(x)) = f^{-1}(f(n(x)) - f(n(x) \vee n(y))) = f^{-1}(g(x) - g(x \wedge y)) = I_{fg}(x,y)$$

**Proposition 10**. If $\perp$ is a $t$ – conorm, then the function of two variables

$$I_{g\perp}(x,y) = g^{-1}(g(1) + g(y) - g(\perp(x,y)))$$

is the fuzzy implication.

**Corollary**. If $f(s) = g(1) - g(s)$, then in accordance with proposition 10, the function of two variables

$$I_{f\perp}(x,y) = f^{-1}(f(y) - f(\perp(x,y)))$$

is the fuzzy implication.

**Example 3**. Let $\perp(x,y) = \max(x,y)$, then in accordance with proposition 10, we have fuzzified Lukasievicz implication

$$I_L(x,y) = \begin{cases} g^{-1}(g(1) - g(x) + g(y)), x > y \\ 1, x \le y \end{cases}$$

or

$$I_L(x,y) = \min\left(1, g^{-1}(g(1) - g(x) + g(y))\right).$$

**Example 4**. Let $\perp(x,y) = \perp_A$, then in accordance with proposition 10, we have fuzzified binary logic implication

$$I_B(x,y) = \begin{cases} n_g(x), y < n_g(x) \\ y, y \ge n_g(x) \end{cases}$$

or

$$I_B(x,y) = \max(n_g(x), y),$$

where $n_g(s) = g^{-1}(g(1) - g(s))$ is the negation function.

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