Fuzzy S* Strong Semicontinuity*

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Abstract: *The aim of this paper is to introduce a new class of function, called fuzzy S* strongly semicontinuous function. Its properties, its relationship with other functions, examples, and preservations of some fuzzy spaces under this function are studied.*

Key words: *Fuzzy topology; Strongly semiopen set; Fuzzy S* strong semicontinuity*

1.Introduction and Preliminaries

The theory of fuzzy continuity for fuzzy topological spaces was introduced by Chang [5]. The concept of fuzzy continuity plays an important role in fuzzy topological spaces. Along this line many weaker forms of fuzzy continuity were introduced [1,2,4,6,7]. In [2], we introduced and studied the concepts of fuzzy strongly semicontinuous function. In this paper, we introduce a new class of function, called fuzzy S* strongly semicontinuous function. It is a weaker form of fuzzy continuity, but it is a stronger form of fuzzy strongly semicontinuity.

Throughout the paper by (X, δ) or simply by X we mean fuzzy topological space(in the Chang's [5] sense), briefly fts. A^o, A^- and A' will denote the interior, closure and complement of fuzzy set A, respectively. A fuzzy set A in X is said to be (1) fuzzy strongly semiopen if there is a fuzzy open set B such that $B \leqslant A \leqslant B^{-o}$, that is, $A \leqslant A^{o-o}$; (2) fuzzy strongly semiclosed if there is a fuzzy closed set B such that $B^{o-} \leqslant A \leqslant B$, that is, $A \leqslant A^{-o-}$ [2]. The family of fuzzy strongly semiopen (resp. strongly semiclosed) sets of a fts X will be denoted by $SSO(X)$ (resp. $SSC(X)$). Let A be a fuzzy set in X. Then $A^{\triangle} = \cup\{B: B \leqslant A, B \in SSO(X)\}$ and $A^{\sim} = \cap\{B: A \leqslant B,$

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$B \in SSC(X)\}$ are called the fuzzy strong semiinterior and fuzzy strong semiclosure of A [2], respectively. Since the union (intersection) of any two fuzzy strongly semiclosed (semiopen) sets need not be a strongly semiclosed (semiopen) set [2], $A \in SSC(X)$ and $B \in SSC(X)$ do not necessarily lead to $A \cup B \in SSC(X)$. Let

$$USSC(X)=\{A \in SSC(X): \text{for each } B \in SSC(X), A \cup B \in SSC(X)\},$$

$$ISSO(X)=\{A \in SSO(X): \text{for each } B \in SSO(X), A \cap B \in SSO(X)\}.$$

Clearly, $\delta \subset ISSO(X) \subset SSO(X)$.

2. Fuzzy S* Strongly Semicontinuous Functions

Definition 2.1. A function $f: (X, \delta) \to (Y, \tau)$ from an fts (X, δ) to another fts (Y, τ) is called fuzzy S* strongly semicontinuous if $f^{-1}(B) \in USSC(X)$ for each $B' \in \tau$.

Theorem 2.2. *A function $f: (X, \delta) \to (Y, \tau)$ is fuzzy S* strongly semicontinuous iff $f^{-1}(B) \in ISSO(X)$ for each $B \in \tau$.*

Proof. This is immediate from Definition 2.1.

Theorem 2.3. *Let $f: (X, \delta) \to (Y, \tau)$ be a fuzzy S* strongly semicontinuous function. Then:*

(1) $f(A^{\sim}) \leqslant (f(A))^{-}$ for each fuzzy set A in X.

(2) $f^{-1}(B))^{\sim} \leqslant f^{-1}(B^{-})$ for each fuzzy set B in Y.

(3) $f^{-1}(B^{o}) \leqslant f^{-1}(B))^{\triangle}$ for each fuzzy set B in Y.

(4) There is a base β for τ such that $f^{-1}(B) \in ISSO(X)$ for each $B \in \beta$.

(5) For each fuzzy point x_{α} in X and each $B \in \tau$ with $f(x_{\alpha}) \in B$, there exists an $A \in ISSO(X)$ such that $x_{\alpha} \in A$ and $f(A) \leqslant B$.

Proof. (1): Let A be a fuzzy set in X. Then $(f(A))^{-}$ is a fuzzy closed set in Y. Since f is fuzzy S* strongly semicontinuous, $f^{-1}((f(A))^{-}) \in USSC(X)$, and

$$A^{\sim} \leqslant (f^{-1}f(A))^{\sim} \leqslant (f^{-1}((f(A))^{-}))^{\sim} = f^{-1}((f(A))^{-}).$$

Thus, $f(A^{\sim}) \leqslant ff^{-1}((f(A))^{-}) \leqslant (f(A))^{-}$.

(2): Let B be a fuzzy set in Y. By (1), $f((f^{-1}(B))^{\sim}) \leqslant (ff^{-1}(B))^{-} \leqslant B^{-}$. Thus,

$$(f^{-1}(B))^{\sim} \leqslant f^{-1}f((f^{-1}(B))^{\sim}) \leqslant f^{-1}(B^{-}).$$

(3): Let B be a fuzzy set in Y. By (2), $f^{-1}(B'^{-}) \geqslant (f^{-1}(B'))^{\sim} = ((f^{-1}(B))')^{\sim}$. From Lemma 1.3 and Theorem 3.3 in [2], we have

$$f^{-1}(B^{o}) = f^{-1}(B'^{-\prime}) = (f^{-1}(B'^{-}))' \leqslant (f^{-1}(B))'^{\sim\prime} = (f^{-1}(B))^{\triangle}.$$

(4): Obvious.

(5): Let f be fuzzy S* strongly semicontinuous, x_{α} be a fuzzy point in X and $B \in \tau$ such that $f(x_{\alpha}) \in B$. Then $x_{\alpha} \in f^{-1}(B)$. Let $A = f^{-1}(B)$, then $A \in ISSO(X)$. We have $f(A)=ff^{-1}(B)\leqslant B$.

Theorem 2.4. *Let $f: (X, \delta)\to(Y, \tau)$ be a fuzzy S* strongly semicontinuous function, and one-to-one and onto. Then $(f(A))^o\leqslant f(A^{\Delta})$ for each fuzzy set A in X.*

Proof. Let f be fuzzy S* strongly semicontinuous and A be any fuzzy set in X. Then $f^{-1}((f(A))^o)\in USSO(X)$. By Theorem 2.3 and the fact that f is one-to-one, we have $f^{-1}((f(A))^o)\leqslant(f^{-1}f(A))^{\Delta})=A^{\Delta}$. Again, since f is onto, we have

$$(f(A))^o=ff^{-1}((f(A))^o)\leqslant f(A^{\Delta}).$$

Proposition 2.5. *If $f: X\to Y$ is a fuzzy S* strongly semicontinuous function and $g: Y\to Z$ is a fuzzy continuous function, then gf is fuzzy S* strongly semicontinuous.*

Theorem 2.6. *Let $f: X_1\to X_2$ and $g: X_3\to X_4$ be fuzzy S* strongly semicontinuous. Then the product $f\times g: X_1\times X_3\to X_2\times X_4$ is fuzzy strongly semicontinuous.*

Proof. Let $B=\cup(A_i\times B_j)$, where the A_i's and B_j's are open sets of X_2 and X_4, respectively. B is a open set of $X_2\times X_4$. Then

$$\begin{aligned}(f\times g)^{-1}(B)&=(f\times g)^{-1}(\cup(A_i\times B_j))\\&=\cup(f\times g)^{-1}(A_i\times B_j)\\&=\cup(f^{-1}(A_i)\times g^{-1}(B_j)).\end{aligned}$$

That $(f\times g)^{-1}(B)$ is a strongly semiopen set follows from Theorem 2.8 and 2.5(1) in [2]. Thus, $f\times g$ is fuzzy strongly semicontinuous.

Theorem 2.7. *Let $p_i: X_1\times X_2\to X_i\ (i=1,2)$ be the projection of $X_1\times X_2$ on X_i. If $f: X\to X_1\times X_2$ is fuzzy S* strongly semicontinuous, then p_if is also fuzzy S* strongly semicontinuous.*

Proof. This follows directly from Proposition 2.5.

Theorem 2.8. *Let $f: X_1\to X_2$ be a function. If the graph $g: X_1\to X_1\times X_2$ of f is fuzzy S* strongly semicontinuous, then f is also fuzzy S* strongly semi- continuous.*

Proof. This follows directly from Theorem 2.7.

3. Examples

Clearly, the following statements are valid:
fuzzy continuity ⇒ S* strong semicontinuity ⇒ strong semicontinuity

None of the converses need to be true. We give the following examples.

Example 3.1. Let $X=[0,1]$ and A,B,C be fuzzy sets in X defined as follows:

$$A(x)=0.2, \quad x\in[0,1];$$
$$B(x)=0.5, \quad x\in[0,1];$$
$$C(x)=0.3, \quad x\in[0,1].$$

Then $\delta=\{0,A,B,1\}$ and $\tau=\{0,C,1\}$ are fuzzy topologies on X. Let f: $(X, \delta)\rightarrow(X, \tau)$ be an identity mapping. Simple computations give $f^{-1}(C)=C\in ISSO(X, \delta)$, $f^{-1}(0)\in ISSO(X, \delta)$, and $f^{-1}(1)\in ISSO(X, \delta)$. Thus, f is fuzzy S* strongly semicontinuous. Clearly, f is not fuzzy continuous.

Example 3.2. Let $X=\{x,y,z\}$, and A,B,C be fuzzy sets in X defined as follows:

$$A(x)=0.3, \quad A(y)=0.2, \quad A(z)=0.7;$$
$$B(x)=0.8, \quad B(y)=0.9, \quad B(z)=0.4;$$
$$C(x)=0.8, \quad C(y)=0.9, \quad C(z)=0.5.$$

Then $\delta=\{0,A,B,A\cap B,A\cup B,1\}$ and $\tau=\{0,C,1\}$ are fuzzy topologies on X. Let f: $(X, \delta)\rightarrow(X, \tau)$ be an identity mapping. We can easily get that $f^{-1}(C)=C$ is a fuzzy strongly semiopen set in (X, δ). Thus, f is fuzzy strongly semicontinuous. Clearly, A and C are fuzzy strongly semiopen sets in (X, δ). Since

$$A\cap C \not\leq (A\cap C)^{o-o}=(A\cap B)^{-o}=(A\cap B)'^{o}=A\cap B,$$

$A\cap C$ is not a fuzzy strongly semiopen set in (X, δ), i.e. $C\notin ISSO(X, \delta)$. Thus, f is not fuzzy S* strongly semicontinuous.

4. Preservation of some topological structures

Definition 4.1 [3,8]. A fuzzy set A is called a connected (strongly connected) set if A cannot be represented as a union of two separated (weakly separated) non-null sets.

Theorem 4.2. *Every fuzzy S* strongly semicontinuous image of a fuzzy strongly connected set is fuzzy connected.*

Proof. Let f: $X\rightarrow Y$ be a fuzzy S* strongly semicontinuous function and A a fuzzy strongly connected set in X. If possible, let $f(A)$ be not fuzzy connected in Y. Then there exist two separated non-null sets B and C in Y such that $f(A)=B\cup C$. Put $E=A\cap f^{-1}(B)$ and $F=A\cap f^{-1}(C)$. Then

$$E\cup F=A\cap(f^{-1}(B)\cup f^{-1}(C))=A\cap \ (f^{-1}f(A))=A,$$

and

$$E^{-}\cap F=(A\cap f^{-1}(B))^{-}\cap(A\cap f^{-1}(C))$$

$\leqslant A^{-} \cap (f^{-1}(B))^{-} \cap A \cap f^{-1}(C)$
$\leqslant A \cap f^{-1}(B^{-}) \cap f^{-1}(C)$
$=A \cap f^{-1}(B^{-} \cap C)$
$=A \cap f^{-1}(0_Y)=0_X.$

Analogously, $E \cap F^{-}=0_X$. Again $E \neq 0_X$, in fact, if $E=0_X$, then $A=F=A \cap f^{-1}(C)$. And so $A \leqslant f^{-1}(C)$, and $f(A) \leqslant F$. Hence, $B \leqslant C$, This is a contradiction. Analogously, $F \neq 0_X$. Thus, A is not a fuzzy strongly connected set in X.

Definition 4.3. A fuzzy topological space (X, δ) is called fuzzy compact (fuzzy I-compact) if for every cover $\{V_a: V_a \in \delta\}$ $(\{V_a: V_a \in ISSO(X)\})$ of X, there exists a finite subcover of X.

Theorem 4.4. *Every surjection fuzzy S* strongly semicontinuous image of a fuzzy I-compact space is fuzzy compact.*

Proof. Let $f: X \rightarrow Y$ be a surjection fuzzy S* strongly semicontinuous function of a fuzzy I-compact space X to a fuzzy topological space Y. Let $\{V_a: a \in J\}$ be a fuzzy open cover of Y. Then $f^{-1}(V_a) \in ISSO(X)$ for each $a \in J$, and $\Phi=\{f^{-1}(V_a): a \in J\}$ is a cover of X. Since X is fuzzy I-compact, there exists a finite subset J_o of J such that $\cup\{f^{-1}(V_a): a \in J_o\}=I_X$. Now

$$I_Y=f(I_X)=f(\cup\{f^{-1}(V_a): a \in J_o\})=\cup\{ff^{-1}(V_a): a \in J_o\} \leqslant \cup\{V_a: a \in J_o\}.$$

Therefore, Y is fuzzy compact.

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