FIXED POINTS FOR FUZZY MAPPINGS IN QUASI-METRIC SPACES

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Abstract: In this paper, we obtain fixed point theorems for fuzzy mappings in Smyth-complete and left K-complete quasi-metric spaces, respectively. The results have improved the fixed theorems of Valentín Gregori and Salvador Romaguera et al.

Keywords: Topology; Fuzzy mapping; Fixed point; Quasi-metric; Smyth-complete; Left K-complete.

1. Preliminaries

Throughout this paper the letter N will denote the set of positive integers. If A is subset of a topological space (X,τ), we will denote by $cl_\tau A$ the closure of A in (X,τ).

A quasi-metric on a (nonempty) set X is a non-negative real-valued function d on $X\times X$ such that, for all x, y, $z\in X$: (i)$d(x,y)=d(y,x)=0$ implies $x=y$; (ii)$d(x,y)\leqslant d(x,z)+d(z,y)$. A quasi-metric space is a pair (X,d) such that X is a (nonempty) set and d is a quasi-metric on X.

Each quasi-metric d on X induces a topology $\tau(d)$ on X which has as a base the family of d-balls $\{B_d(x,r):x\in X,r>0\}$, where $B_d(x,r)=\{y\in X:d(x,y)<r\}$. Each quasi-metric d on X also induces a conjugate quasi-metric d^{-1}, defined by $d^{-1}(x,y)=d(y,x)$.

By d^s we denote the metric $d\vee d^{-1}$ $(i.e. d^s(x,y)=\max\{d(x,y),d(y,x)\}$, for all $x,y\in X)$.

Definition 1. A sequence $\{x_n\}_{n\in N}$ in a quasi-metric space (X,d) is called left K-Cauchy if for each $\varepsilon>0$ there is an $n_\varepsilon\in N$ such that $d(x_n,x_m)<\varepsilon$ for all $n,m\in N$ such that $n_\varepsilon\le n\le m$.

The quasi-metric space (X,d) is left K-complete[3], provided that every left K-Cauchy sequence in (X,d) is convergent with respect to the topology $\tau(d)$. (X,d) is Smyth-complete provided that every left K-Cauchy sequence in (X,d) is convergent in the metric space (X,d^s), (See[5,6]). Clearly, every Smyth-complete quasi-metric space is left K-complete.

A fuzzy set in the quasi-metric space (X,d) is a function from X into the unit interval $[0,1]$. If A is a fuzzy set in X, then, for each $x\in X$, the number $A(x)$ is called the grade of membership of x in A. The r-level of A, denoted by A_r, is defined by $A_r=\{x\in X: A(x)\ge r\}$ if $r\in(0,1]$, and $A_0=\overline{\{x\in X:A(x)>0\}}$, where the closure is taken in (X, d^s).

Definition 2. A fuzzy set A in the quasi-metric space (X,d) will be called an approximate quantity if for each $r \in [0,1]$, A_r is compact in (X, d^s) and $\sup_{x \in X} A(x)=1$.

By $W(X)$ we will denote the collection of all approximate quantities in the quasi-metric space (X,d).

Definition 3. Let (X,d) is a quasi-metric space, $A, B \in W(X)$, $r \in [0,1]$. Then we define

$P_r(A,B) = \inf\{d(x,y) : x \in A_r, y \in B_r\}$, $P(A,B) = \sup\{P_r(A,B) : r \in [0,1]\}$

$D_r(A,B) = H_d(A_r, B_r)$, where H is the Hausdorf distance;

$D(A,B) = \sup\{D_r(A,B) : r \in [0,1]\}$,

We will use the following lemmas due [1] and proposition due[2].

Lemma 1. Let (X,d) be a quasi-metric space. Then, for each $A \in W(X)$ there exists $p \in X$ such that $A(p) = 1$.

Lemma 2. Let (X,d) be a quasi-metric space, $A, B \in W(X)$ and $x \in A_1$ (such an x exists by Lemma 1). Then there is $y \in B_1$ such that $d(x,y) \le D_1(A,B)$.

Lemma 3. Let (X,d) be a quasi-metric space and let $A, B \in W(X)$. Then, $p(A,B) = p_1(A,B)$.

Lemma 4. Let (X,d) be a quasi-metric space, $A \in W(X)$ and $y \in A_1$. Then, for each $x \in X$, $p(x,A) \le d(x,y)$.

Lemma 5. Let (X,d) be a quasi-metric space and let $A \in W(X)$. Then, for each $x, y \in X$ and each $r \in [0,1]$, $p_r(x,A) \le d(x,y) + p_r(y,A)$.

Lemma 6. Let (X,d) be a quasi-metric space, $A \in W(X)$ and $x \in A_1$. Then, for each $B \in W(X)$ and each $r \in [0,1]$, $p_r(x,B) \le D_r(A,B)$

Lemma 7. Let (X,d) be a quasi-metric space and $A \in W(X)$. If $p(x,A) = 0$, then there is $y \in cl_{\tau(d^{-1})}\{x\}$ such that $A(y) = 1$.

Proposition 1. Let (X, d) be a quasi-pseudo-metric space and let $\{x_n\}_{n \in N}$ be a sequence in X such that $\sum_{i=1}^{\infty} d(x_i, x_{i+1}) < \infty$. Then $\{x_n\}_{n \in N}$ is a left K-Cauchy sequence in (X,d).

Definition 4. A fuzzy mapping on a quasi-metric space (X,d) is a function F defined on X, which satisfies the following two conditions:

(i) $F(x) \in W(X)$ for all $x \in X$.

(ii) If z and a are points of X such that $(F(z))(a) = 1$ and $p(a, F(a)) = 0$, then $(F(a))(a) = 1$.

Definition 5. We say that a fuzzy mapping F on a quasi-metric space (X,d) has a fixed point if there exists $a \in X$ such that $(F(a))(a) = 1$.

2. Main Results

Theorem 1. Let *(X,d)* be a Smyth-complete quasi-metric space, let *F* and *G* be fuzzy mappings from *X* into *W(X)*. If there exists a constant *h*, $0 \le h < 1$, such that for each $x, y \in X$.

$$D(F(x), G(y)) \le h \max\{d(x,y), p(x,F(x)), p(y,G(y)), (p(x,G(y)) + p(y,F(x)))/2\}, \quad (1)$$

Then, *F* and *G* each have a fixed point.

Proof. Let $x_0 \in X$ .By Lemma 1, there exists an $x_1 \in X$ such that $(F(x_0))(x_1) = 1$. By Lemma2 there exists an $x_2 \in X$ such that $(G(x_1))(x_2) = 1$ and $d(x_1, x_2) \le D_1(F(x_0), G(x_1))$. Again we can find an $x_3 \in X$ such that $(F(x_2))(x_3) = 1$ and $d(x_2, x_3) \le D_1(G(x_1), F(x_2))$.

Continuing in this way we can produce a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that

$$(F(x_{2n}))(x_{2n+1}) = 1, \quad (G(x_{2n+1}))(x_{2n+2}) = 1, n = 0,1,2,\cdots$$

$$d(x_{2n+1}, x_{2n+2}) \le D_1(F(x_{2n}), G(x_{2n+1})), n = 1,2,\cdots \quad (2)$$

$$d(x_{2n}, x_{2n+1}) \le D_1(G(x_{2n-1}), F(x_{2n})), n = 1,2,\cdots. \quad (3)$$

We then have,

$$d(x_1, x_2) \le D_1(F(x_0), G(x_1)) \le D(F(x_0), G(x_1))$$
$$\le h \max\{d(x_0, x_1), P(x_0, F(x_0)), P(x_1, G(x_1)), (P(x_0, G(x_1)) + P(x_1, F(x_0)))/2\}.$$

So, by Lemma 4,

$$d(x_1, x_2) \le h \max\{d(x_0, x_1), d(x_1, x_2), d(x_0, x_2)/2\} = hd(x_0, x_1).$$

Similarly, $d(x_2, x_3) \le hd(x_1, x_2)$, so $d(x_2, x_3) \le h^2 d(x_0, x_1)$.

Next we show by induction that

$$d(x_n, x_{n+1}) \le h^n d(x_0, x_1), \quad n = 1,2,\cdots \quad (4)$$

In fact, by the assumptions it is obvious that (4) is true for n=1. Suppose that (4) is true for *n*=*k*; we prove that it remains true for *n*=*k*+1.

If *k* is even, then from conditions (2) and (3) we have

$$d(x_{k+1}, x_{k+2}) \le D_1(F(x_k), G(x_{k+1})) \le D(F(x_k), G(x_{k+1}))$$
$$\le k \max\{d(x_k, x_{k+1}), P(x_k, F(x_k)), P(x_{k+1}, G(x_{k+1})), ((P(x_k, G(x_{k+1})) + P(x_{k+1}, F(x_k)))/2\}$$

By Lemma 4, $P(x_k, F(x_k)) \le d(x_k, x_{k+1})$, $P(x_{k+1}, G(x_{k+1})) \le d(x_{k+1}, x_{k+2})$, $P(x_k, G(x_{k+1})) \le d(x_k, x_{k+2})$, $P(x_{k+1}, F(x_k)) \le d(x_{k+1}, x_{k+1})$. Hence,

$$d(x_{k+1}, x_{k+2}) \le h \max\{d(x_k, x_{k+1}), d(x_k, x_{k+1}), d(x_{k+1}, x_{k+2}), (d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + d(x_{k+1}, x_{k+1}))/2\}$$
$$= hd(x_k, x_{k+1}) \le h \cdot h^n d(x_0, x_1) = h^{n+1} d(x_0, x_1). \quad (5)$$

If *k* is odd we can similarly prove that (5) remains true.This completes the proof of (4).

It follows from proposition 1, that $\{x_n\}_{n=1}^{\infty}$ is left K-Cauchy sequence in (X,d). Hence, there exists a unique point $z \in X$ such that $d^s(z,x_n) \to 0 \quad (n \to \infty)$.

Applying inequality (1) we get

$$D(F(x_{2n}),G(z)) \le h\max\{d(x_{2n},z),P(x_{2n},F(x_{2n})),\ P(z,G(z)), (P(x_{2n},G(z))+P(z,F(x_{2n})))/2\}.$$

Now, by Lemma 5, we have

$$P_1(z,G(z)) \le d(z,x_{2n+1}) + P_1(x_{2n+1},G(z)) \text{ for all } n \in N.$$

So, by Lemma 3 and 6,

$$P(z,G(z)) \le d(z,x_{2n+1}) + D_1(F(x_{2n}),G(z)) \le d(z,x_{2n+1}) + D(F(x_{2n}),G(z)) \qquad (6)$$

Moreover, by Lemma 4, $P(x_{2n},F(x_{2n})) \le d(x_{2n},x_{2n+1})$,

$P(z,F(x_{2n})) \le d(z,x_{2n+1})$, because $x_{2n+1} \in (F(x_{2n}))_1$.

By Lemma 3, 5 and 6,

$$P(x_{2n},G(z)) \le d(x_{2n},x_{2n+1}) + P(x_{2n+1},G(z)) \le d(x_{2n},x_{2n+1}) + D(F(x_{2n}),G(z))$$

from which it follows that

$$D(F(x_{2n}),G(z)) \le h\max\{d(x_{2n},z),d(x_{2n},x_{2n+1}),d(z,x_{2n+1})+D(F(x_{2n}),G(z)), (d(x_{2n},x_{2n+1})+D(F(x_{2n}),G(z))+d(z,x_{2n+1}))/2\} = h\Phi(F,G),$$

where, $\Phi(F,G) = \max\{d(x_{2n},z),d(x_{2n},x_{2n+1}),d(z,x_{2n+1})+D(F(x_{2n}),G(z)), (d(x_{2n},x_{2n+1})+D(F(x_{2n}),G(z))+d(z,x_{2n+1}))/2\}$.

Now there are several cases:

Case1. For each $n \in N$, $\Phi(F,G) = d(x_{2n},z)$.

Case2. For each $n \in N$, $\Phi(F,G) = d(x_{2n},x_{2n+1})$

Case3. For each $n \in N$, $\Phi(F,G) = d(z,x_{2n+1}) + D(F(x_{2n}),G(z))$.

Case4. For each $n \in N$,

$$\Phi(F,G) = (d(x_{2n},x_{2n+1}) + D(F(x_{2n}),G(z)) + d(z,x_{2n+1}))/2.$$

In case1, from $d^s(z,x_n) \to 0$, it follows that $D(F(x_{2n}),G(z)) \to 0$. By (6), $P(z,G(z)) = 0$.

In case2, since $\{x_n\}_{n=1}^{\infty}$ is left K-Cauchy sequence, we obtain that $d(x_{2n},x_{2n+1}) \to 0$. So,

$D(F(x_{2n}),G(z)) \to 0$. From (6) and the fact that $d^s(z,x_n) \to 0$, we deduce that $P(z,G(z)) = 0$.

In case3, we obtain that $(1-h)D(F(x_{2n}),G(z)) \le hd(z,x_{2n+1})$. Hence, $D(F(x_{2n}),G(z)) \to 0 (n \to \infty)$. Again by (6), $P(z,G(z)) = 0$.

Finally, in case4, we obtain

$$(2-h)D(F(x_{2n}),G(z)) \le h(d(x_{2n},x_{2n+1}) + d(z,x_{2n+1}))$$

Hence, $D(F(x_{2n}),G(z)) \to 0 (n \to \infty)$. By (6), $P(z,G(z)) = 0$

Therefore, by Lemma 7 there exists $a_1 \in cl_{\tau(d^{-1})}\{z\}$ such that $(G(z))(a_1)=1$. We shall prove that a_1 is a fixed point of *F*.

Indeed, since $(G(z))(a_1)=1, P(a_1,G(z))=0$.

We also have that $d(z,a_1)=0$, and $P(z,G(z))=0$, as we have proved.

Applying inequality (1) we get

$$D(F(a_1),G(z)) \le h\max\{d(a_1,z),P(a_1,F(a_1)),P(z,G(z)),$$
$$(P(a_1,G(z))+P(z,F(a_1)))/2\}.$$

It follows, from the preceding relations, that

$$D(F(a_1),G(z)) \le h\max\{P(a_1,F(a_1)),P(z,F(a_1))/2\}. \qquad (7)$$

By Lemma 1 and 2, there exists $c \in X$ such that

$(F(a_1))(c)=1$ and $d(a_1,c) \le D_1(F(a_1),G(z)) \le D(F(a_1),G(z))$.

Thus, by Lemma 4, $P(a_1,F(a_1)) \le d(a_1,c) \le D(F(a_1),G(z))$

and $P(z,F(a_1)) \le d(z,c)$, because $(F(a_1))(c)=1$.

So, $P(z,F(a_1)) \le d(z,a_1)+d(a_1,c) \le D(F(a_1),G(z))$.

Then, by hypothesis (7), we have

$D(F(a_1),G(z)) \le hD(F(a_1),G(z))$. We conclude that $D(F(a_1),G(z))=0$ and, consequently, $d(a_1,c)=0$. Thus, $a_1=c$, and $(F(a_1))(a_1)=1$. By Definition 4, a_1 is a fixed point of *F*. Similarly, we can shown *G* has a fixed point. This completes the proof of Theorem 1.

Theorem 2. Let *(X,d)* be a left *K*-complete quasi-metric space and *F* and *G* be fuzzy mappings from *X* into *W(X)* satisfying

$D(F(x),G(y)) \le h[P(x,F(x))P(y,G(y))]^{1/2}$

for all $x,y \in X$ and $0<h\le 1$. Then, *F* and *G* each have a fixed point.

Proof. Let $x_0 \in X$. As in the first part of the proof of Theorem 1, we can construct a sequence $\{x_n\}_{n=1}^{\infty}$ in *X*, such that

$(F(x_{2n}))(x_{2n+1})=1$, $(G(x_{2n+1}))(x_{2n+2})=1, n=0,1,2,\cdots$.

and $d(x_{2n+1},x_{2n+2}) \le D_1(F(x_{2n}),G(x_{2n+1})), n=1,2,\cdots$,

$d(x_{2n},x_{2n+1}) \le D_1(G(x_{2n-1}),F(x_{2n})), n=1,2,\cdots$.

By Lemma 2, we have

$$d(x_1,x_2) \le D_1(F(x_0),G(x_1)) \le D(F(x_0),G(x_1))$$
$$\le \frac{1}{\sqrt{h}}D(F(x_0),G(x_1)) \le \sqrt{h}[P(x_0,F(x_0))P(x_1,G(x_1))]^{\frac{1}{2}}.$$

By Lemma 4, we have

$$d(x_1,x_2) \le \sqrt{h}[d(x_0,x_1)d(x_1,x_2)]^{\frac{1}{2}}.$$

So, $d(x_1,x_2) \le hd(x_0,x_1)$.

Similarly, we can find $x_3 \in X$ such that $(F(x_2))(x_3)=1$ and $d(x_2,x_3) \le D_1(G(x_1),F(x_2)) \le D(G(x_1), F(x_2))$

$$\le \frac{1}{\sqrt{h}} D(G(x_1),F(x_2)) \le \sqrt{h}[P(x_1,G(x_1))P(x_2,F(x_2)]^{\frac{1}{2}}.$$

By Lemma 4, we have

$$d(x_2,x_3) \le \sqrt{h}[d(x_1,x_2)d(x_2,x_3)]^{\frac{1}{2}}. \text{ So, } d(x_2,x_3) \le hd(x_1,x_2) \le h^2 d(x_0,x_1).$$

In general, $d(x_{n+1},x_{n+2}) \le hd(x_n,x_{n+1})$ for $n=0,1,2\cdots$, where $(F(x_{2n-2}))(x_{2n-1})=1$, $(G(x_{2n-1}))(x_{2n})=1$ are such that $d(x_{2n-1},x_{2n}) \le \frac{1}{\sqrt{h}} D(F(x_{2n-2}),G(x_{2n-1}))$ and consequently we obtain $d(x_n,x_{n+1}) \le h^n d(x_0,x_1)$ for all $n \in N$.

It follows from proposition 1, that $\{x_n\}_{n=1}^{\infty}$ is a left *K*-Cauchy sequence in (X,d).

Hence, there exists a point $z \in X$, such that $d(z,x_n) \to 0$.

By Lemma 3, 5 and 6 it follows, similar to the proof of Theorem 1, that

$$\begin{aligned} P(z,F(z)) &\le d(z,x_{2n}) + P_1(x_{2n},F(z)) \le d(z,x_{2n}) + D_1(G(x_{2n-1}),F(z)) \\ &\le d(z,x_{2n}) + D(G(x_{2n-1}),F(z)) \end{aligned} \tag{8}$$

for all $n \in N$. But by Lemma 4 get

$$D(G(x_{2n-1}),F(z)) \le h[p(x_{2n-1},G(x_{2n-1}))P(z,F(z))]^{\frac{1}{2}} \le h[d(x_{2n-1},x_{2n})P(z,F(z))]^{\frac{1}{2}}.$$

From (8) we get

$$P(z,F(z)) \le d(z,x_{2n}) + h[d(x_{2n-1},x_{2n})P(z,F(z))]^{\frac{1}{2}}.$$

When $n \to \infty$, we have $P(z,F(z)) \le 0$, which give $P(z,F(z))=0$.

By Lemma 7, there exists $a_1 \in cl_{\tau(d^{-1})}\{z\}$ such that $(F(z))(a_1)=1$.

Similarly

$$\begin{aligned} P(z,G(z)) &\le d(z,x_{2n+1}) + P_1(x_{2n+1},G(z)) \le d(z,x_{2n+1}) + D_1(F(x_{2n}),G(z)) \\ &\le d(z,x_{2n+1}) + D(F(x_{2n}),G(z)) \end{aligned} \tag{9}$$

for all $n \in N$. But by Lemma 4 get

$$D(F(x_{2n}),G(z)) \le h[P(x_{2n},F(x_{2n}))P(z,G(z))]^{\frac{1}{2}} \le h[d(x_{2n},x_{2n+1})P(z,G(z))]^{\frac{1}{2}}.$$

From (9) we get

$$P(z,G(z)) \le d(z,x_{2n+1}) + h[d(x_{2n},x_{2n+1})P(z,G(z))]^{\frac{1}{2}}.$$

When $n \to \infty$, we have $P(z,G(z)) \le 0$, which give

$$P(z,G(z))=0 \tag{10}$$

By Lemma 7, there exists $a_2 \in cl_{\tau(d^{-1})}\{z\}$ such that $(G(z))(a_2)=1$.

We shall prove that a_2 is a fixed point of *F* and a_1 is a fixed Point of *G*.

Indeed, since

$D(F(z),G(a_1)) \le h[P(z,F(z))P(a_1,G(a_1))]^{\frac{1}{2}}$, it follows that

$D(F(z),G(a_1)) = 0$, because $p(z,F(z)) = 0$.

On the other hand, by Lemma 1 and 2, there exists $b \in X$ such that $(G(a_1))(b) = 1$ and $d(a_1,b) \le D_1(F(z),G(a_1)) \le D(F(z),G(a_1))$.

Therefore, $d(a_1,b) = 0$. Thus, $b = a_1$, and $(G(a_1))(a_1) = 1$.

By Definition 4. a_1 is a fixed point of *G*.

Similarly, we can shown a_2 is a fixed point of *F*. This completes the proof of Theorem 2.

If (X,d) is a quasi-metric space and $A, B \in W(X)$, we define, as in the metric case,

$$\delta(A,B) = \sup_{r\in[0,1]} \delta_r(A_r,B_r), \text{ where } \delta_r(A_r,B_r) = \sup\{d(a,b) : a \in A_r \text{ and } b \in B_r\}.$$

Clearly, $D(A,B) \le \delta(A,B)$ for all $A, B \in W(X)$.

Hence, we immediately deduce from Theorem 2 the following:

Corollary. Let (X,d) be a left *K*-complete quasi-metric space and *F* and *G* be fuzzy mappings from *X* into *W(X)* satisfying the following condition: For any *x,y* in *X*

$$\delta(F(x),G(y)) \le h[P(x,F(x))P(y,G(y))]^{\frac{1}{2}}, \text{ where } 0 < h < 1.$$

Then, *F* and *G* each have a fixed point.

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