Lacunary statistical convergence of difference sequences of fuzzy numbers

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Abstract : In this paper we introduce the spaces of bounded and convergent difference sequences of fuzzy numbers and the space of lacunary statistcally convergent difference sequences of fuzzy numbers and give some relations related to these sequence spaces.

Key words and phrases: Fuzzy number, statistical convergence, difference sequence, lacunary sequence.

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh [19]. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [11]. Matloka show that every convergent sequence of fuzzy numbers is bounded. Later on sequences of fuzzy numbers have been discussed by Nanda [13], Nuray and Savas [15], Nuray [16], Kwon [8], Savas [17], Bilgin [2], Basarir and Mursaleen [1,12], Fang and Huang [4] and many others.

The natural density of a set A of positive integers is defined by

$$\delta(A) = \lim_n \frac{1}{n} |\{k \leq n : k \in A\}|$$

where $|\{k \leq n : k \in A\}|$ denotes the number of elements of $A \subseteq \mathbb{N}$ does not exceeding n [14]. It is clear that any finite subset of $\mathbb{N}$ have zero natural density and $\delta(A^c) = 1 - \delta(A)$. If a property $P(k)$ holds for all $k \in A$ with $\delta(A) = 1$, we say that P holds for almost all k, that is a.a.k.

The concept of statistical convergence was introduced by Fast [5]. Schoenberg [18] studied statistical convergence as a summability method and listed some of elementary properties of statistical convergence.

A sequence (x_k) is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\}) = 0$. In this case we write $S - \lim x_k = L$.

Let $C(\mathbb{R}^n) = \{A \subset R^n : A \text{ compact and convex}\}$. The space $C(\mathbb{R}^n)$ has linear structure induced by the operations $A + B = \{a + b : a \in A,\ b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The Hausdorff distance between A and B of $C(\mathbb{R}^n)$ is defined as

$$\delta_\infty(A, B) = \max\left\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\right\}.$$

It is well kwon that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete (not separable) metric space.

A fuzzy number is a function X from $\mathbb{R}^n$ to $[0,1]$ which is normal, fuzzy convex, upper semi-continuous and the closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the $\alpha-$ level set $[X]^\alpha = \{x \in \mathbb{R}^n : X(x) \geq \alpha\}$ is a nonempty compact convex subset of $\mathbb{R}^n$, as is support X^0. Let $L(\mathbb{R}^n)$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R}^n)$ induces the addition $X+Y$ and scalar multiplication λX, $\lambda \in \mathbb{R}$, in terms of $\alpha-$ level sets, by

$$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha \qquad [\lambda X]^\alpha = \lambda [X]^\alpha$$

for each $0 \le \alpha \le 1$.

Define, for each $1 \le q < \infty$,

$$d_q(X,Y) = \left(\int_0^1 \delta_\infty (X^\alpha, Y^\alpha)^q \, d\alpha \right)^{1/q}$$

and $d_\infty = \sup\limits_{0 \le \alpha \le 1} \delta_\infty (X^\alpha, Y^\alpha)$, where δ_∞ is the Hausdorff metric. Clearly $d_\infty (X,Y) = \lim\limits_{q \to \infty} d_q(X,Y)$ with $d_q \le d_r$ if $q \le r$. Moreover d_q is a complete, separable and locally compact metric space [3]. Throughout the paper, d will denote d_q with $1 \le q \le \infty$.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will denote by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. Lacunary sequence have been discussed in [2,6,7,9,10,15,16].

2. Definitions

Definition 2.1. Let $X = (X_k)$ be a sequence of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to be Δ^2-bounded if the set $\{\Delta^2 X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded and Δ^2- convergent to the fuzzy number X_0 , written as $\lim\limits_k \Delta^2 X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer n_0 such that $d(\Delta^2 X_k, X_0) < \varepsilon$ for $n > n_0$, where $\Delta^2 X = (\Delta X_k - \Delta X_{k+1})$ and $\Delta X = (X_k - X_{k+1})$. Let $m(\Delta^2)$ and $c(\Delta^2)$ denote the set of all Δ^2- bounded sequences and all Δ^2- convergent sequences of fuzzy numbers, respectively.

Definition 2.2. Let $\theta = (k_r)$ be a lacunary sequence and let $X = (X_k)$ be a sequence of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy number is said to be lacunary Δ^2-statistically convergent to a fuzzy numbers X_0 if for every $\varepsilon > 0$

$$\lim_{r \to \infty} h_r^{-1} \left| \{ k \in I_r : d(\Delta^2 X_k, X_0) \ge \varepsilon \} \right| = 0.$$

In this case we write $X_k \to X_0 \left(S_\theta (\Delta^2) \right)$ or $S_\theta - \lim \Delta^2 X_k = X_0$.

The set of all lacunary Δ^2-statistically convergent sequences is denoted by $S_\theta (\Delta^2)$.

In the special case $\theta = (2^r)$, we shall write $S(\Delta^2)$ instead of $S_\theta (\Delta^2)$.

Definition 2.3. Let $\theta = (k_r)$ be a lacunary sequence and let $X = (X_k)$ be a sequence of fuzzy numbers. The sequence X is said to be lacunary strongly Δ_p^2-summable if there is a fuzzy number X_0 such that

$$\lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} d(\Delta^2 X_k, X_0)^p = 0.$$

In this case we write $X_k \to X_0 \left(N_\theta (\Delta_p^2) \right)$ or $N_{\theta p} - \lim \Delta^2 X_k = X_0$. We shall use $N_\theta (\Delta_p^2)$ to denote the set of all lacunary strongly Δ_p^2-Cesàro convergent sequences of fuzzy numbers.

In the special cases $\theta = (2^r)$ and $p = 1$ we shall write $\left|\sigma\left(\Delta_p^2\right)\right|$ and $N_\theta\left(\Delta^2\right)$ instead of $N_\theta\left(\Delta_p^2\right)$, respectively.

It can be shown that if a sequence of fuzzy numbers Δ^2-convergent to fuzzy number X_0, then it is Δ^2-statitically convergent to fuzzy number X_0, but the converse does not hold. For example define $X = (X_k)$ such that

$$\Delta^2 X_k = \begin{cases} A, & \text{if } k = n^2 \ , n = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

where A is a fixed fuzzy number. Then $S - \lim \Delta^2 X_k = A$ and $\lim \Delta^2 X_k \neq A$.

3. Main Results

In this section we prove some results relating to the above sequence spaces.

Theorem 3.1 Let $\{X_k\}$ and $\{Y_k\}$ be sequences of fuzzy numbers.

i) If $S_\theta - \lim \Delta^2 X_k = X_0$ and $c \in \mathbb{R}$, then $S_\theta-\ \lim c\Delta^2 X_k = c\ X_0$

ii) If $S_\theta - \lim \Delta^2 X_k = X_0$ and $S_\theta-\ \lim \Delta^2 Y_k = Y_0$, then $S_\theta - \lim\left(\Delta^2 X_k + \Delta^2 Y_k\right) = X_0 + Y_0$.

Proof. i) Let $\alpha \in [0, 1]$ and $c \in \mathbb{R}$. Let $\Delta^2 X_k^\alpha$, $\Delta^2 Y_k^\alpha$, X_0^α and Y_0^α be α level sets of $\Delta^2 X_k$, $\Delta^2 Y_k$, X_0 and Y_0, respectively. Since $\delta_\infty\left(c\Delta^2 X_k^\alpha, cX_0^\alpha\right) = |c|\, \delta_\infty\left(\Delta^2 X_k, X_0^\alpha\right)$, we have

$$d\left(c\Delta^2 X_k, cX_0\right) = |c|\, d\left(\Delta^2 X_k, X_0\right)$$

For a given $\varepsilon > 0$ we have

$$\frac{1}{h_r}\left|\left\{k \in I_r : d\left(c\Delta^2 X_k, cX_0\right) \geq \varepsilon\right\}\right| \leq \frac{1}{h_r}\left|\left\{k \in I_r : d\left(\Delta^2 X_k, X_0\right) \geq \frac{\varepsilon}{|c|}\right\}\right|.$$

Hence $S_\theta-\ \lim c\Delta^2 X_k = c\ X_0$.

ii) Suppose that $S_\theta - \lim \Delta^2 X_k = X_0$ and $S_\theta - \lim \Delta^2 Y_k = Y_0$. Firstly, we have

$$\begin{aligned} \delta_\infty\left(\Delta^2 X_k^\alpha + \Delta^2 Y_k^\alpha, X_0^\alpha + Y_0^\alpha\right) &\leq \delta_\infty\left(\Delta^2 X_k^\alpha + \Delta^2 Y_k^\alpha, \Delta^2 Y_k^\alpha + X_0^\alpha\right) \\ &\quad + \delta_\infty\left(\Delta^2 Y_k + X_0^\alpha, X_0^\alpha + Y_0^\alpha\right) \\ &= \delta_\infty\left(\Delta^2 X_k^\alpha, X_0^\alpha\right) + \delta_\infty\left(\Delta^2 Y_k^\alpha, Y_0^\alpha\right). \end{aligned}$$

By Minkowski's inequality we get

$$d\left(\Delta^2 X_k + \Delta^2 Y_k, X_0 + Y_0\right) \leq d\left(\Delta^2 X_k, X_0\right) + d(\Delta^2 Y_k, Y_0)$$

Therefore given $\varepsilon > 0$ we have,

$$\begin{aligned} & \frac{1}{h_r}\left|\left\{k \in I_r : d\left(\Delta^2 X_k + \Delta^2 Y_k, X_0 + Y_0\right) \geq \varepsilon\right\}\right| \\ \leq\ & \frac{1}{h_r}\left|\left\{k \in I_r : d\left(\Delta^2 X_k, X_0\right) + d(\Delta^2 Y_k, Y_0) \geq \varepsilon\right\}\right| \\ \leq\ & \frac{1}{h_r}\left|\left\{k \in I_r : d(\Delta^2 X_k, X_0) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{h_r}\left|\left\{k \in I_r : d(\Delta^2 Y_k, Y_0) \geq \frac{\varepsilon}{2}\right\}\right|. \end{aligned}$$

Hence $S_\theta-\ \lim\left(\Delta^2 X_k + \Delta^2 Y_k\right) = X_0 + Y_0$.

Theorem 3.2. $c(\Delta^2)$ and $m(\Delta^2)$ are complete metric spaces with the metric

$$\rho(X,Y)_\Delta = d(X_1,Y_1) + d(X_2,Y_2) + \sup_n d\left(\Delta^2 X_n, \Delta^2 Y_n\right)$$

and $N_\theta\left(\Delta_p^2\right)$ is a complete metric space with the metric

$$\delta(X,Y)_\Delta = d(X_1,Y_1) + d(X_2,Y_2) + \sup_r \left(h_r^{-1}\sum_{i\in I_r} d\left(\Delta^2 X_i, \Delta^2 Y_i\right)^p\right)^{\frac{1}{p}}, \ 1 \le p < \infty.$$

Proof. We shall prove only for the space $N_\theta\left(\Delta_p^2\right)$. The others can be proved by the same way. Let (X^n) be a Cauchy sequence in $N_\theta\left(\Delta_p^2\right)$, where $X^n = (X_i^n)_i = (X_1^n, X_2^n, ...) \in N_\theta\left(\Delta_p^2\right)$ for each $n \in N$. Then

$$\delta(X^n,X^m)_\Delta = d(X_1^n,X_1^m) + d(X_2^n,X_2^m) + \sup_r \left(h_r^{-1}\sum_{i\in I_r} d\left(\Delta^2 X_i^n, \Delta^2 X_i^m\right)^p\right)^{\frac{1}{p}} \to 0, \text{ as } m,n \to \infty.$$

Hence we obtain

$$d(X_i^n, Y_i^m) \to 0, \text{ as } m,n \to \infty, \text{for each } i \in N.$$

Therefore $(X_i^n)_i = \left(X_i^1, X_i^2, ...\right)$ is a Cauchy sequence in $L(R)$. Since $L(R)$ is complete, it is convergent

$$\lim_n X_i^n = X_i$$

say, for each $i \in N$. Since (X^n) is a Cauchy sequence, for each $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that

$$\delta(X^n,X^m)_\Delta < \varepsilon \text{ for all } m,n \ge n_0.$$

Hence $d(X_1^n,X_1^m) \le \varepsilon$, $d(X_2^n,X_2^m) \le \varepsilon$ and $h_r^{-1}\sum_{i\in I_r} d\left(\Delta^2 X_i^n, \Delta^2 X_i^m\right)^p < \varepsilon^p$ for all $r \in N$ and for all $m,n \ge n_0$. So we have

$$\lim_m d(X_1^n,X_1^m) = d(X_1^n,X_1) < \varepsilon, \qquad \lim_m d(X_2^n,X_2^m) = d(X_2^n,X_2) < \varepsilon$$

and

$$\lim_m h_r^{-1}\sum_{i\in I_r} d\left(\Delta^2 X_i^n, \Delta^2 X_i^m\right)^p = h_r^{-1}\sum_{i\in I_r} d\left(\Delta^2 X_i^n, \Delta^2 X_i\right)^p < \varepsilon^p$$

for all $r \in N$ and $n \ge n_0$. This implies that $\delta(X^n,X)_\Delta < 3\varepsilon$, that is $X^n \to X$, $n \to \infty$, where $X = (X_i)$. Since

$$h_r^{-1}\sum_{i\in I_r} d\left(\Delta^2 X_i, X_0\right)^p \le 2^p \left\{h_r^{-1}\sum_{i\in I_r} d\left(\Delta^2 X_i^N, X_0\right)^p + h_r^{-1}\sum_{i\in I_r} d\left(\Delta^2 X_i^N, \Delta^2 X_i\right)^p\right\} \to 0.$$

as $n \to \infty$, we obtain $X \in N_\theta\left(\Delta_p^2\right)$. Therefore $N_\theta\left(\Delta_p^2\right)$ is a complete metric space.

It can be shown that $N_\theta\left(\Delta_p^2\right)$ is a complete metric space with the metric

$$\delta\left(X,Y\right)_{\Delta'} = d\left(X_1,Y_1\right)^p + d\left(X_2,Y_2\right)^p + \sup_r h_r^{-1}\sum_{i\in I_r} d\left(\Delta^2 X_i, \Delta^2 Y_i\right)^p, \text{ for } 0<p<1.$$

Theorem 3.3. If $\{X_k\} \in S\left(\Delta^2\right) \cap S_\theta\left(\Delta^2\right)$, then we have $S_\theta - \lim \Delta^2 X_k = S - \lim \Delta^2 X_k$.

Proof. Suppose that $S - \lim \Delta^2 X_k = X_0$ and $S_\theta - \lim \Delta^2 X_k = X_0'$ and $X_0 \neq X_0'$. Then, we have $d\left(X_0, X_0'\right) > 0$. For $\left(d\left(X_0,X_0'\right)\right)/2 > \varepsilon > 0$, we have

$$\lim_n \frac{1}{n}\left|\left\{k\leq n : d\left(\Delta^2 X_k, X_0\right) \geq \varepsilon\right\}\right| = 1.$$

Now consider the k_m-th term of the statistical limit expression $\frac{1}{n}\left|\left\{k \leq n : d\left(\Delta^2 X_k, X_0\right) \geq \varepsilon\right\}\right|$:

$$\frac{1}{k_m}\left|\left\{k \in \cup_{k=1}^m I_r : d\left(\Delta^2 X_k, X_0'\right) \geq \varepsilon\right\}\right| = \frac{1}{k_m}\sum_{r=1}^m \left|\left\{k \in I_r : d\left(\Delta^2 X_k, X_0'\right) \geq \varepsilon\right\}\right| = \frac{1}{\sum_{r=1}^m h_r}\sum_{r=1}^m h_r t_r \tag{1}$$

where $t_r = \frac{1}{h_r}\left|\left\{k\in I_r : d\left(\Delta^2 X_k, X_0'\right)\geq\varepsilon\right\}\right| \to 0$, since $S_\theta - \lim \Delta^2 X_k = X_0'$. Since θ is a lacunary and $\frac{1}{\sum_{r=1}^m h_r}\sum_{r=1}^m h_r t_r$ is a regular weighted mean transform of t, it goes to zero as $m \to \infty$. Since (1) is a subsequence of $\left\{\frac{1}{n}\left|\left\{k\leq n : d\left(\Delta^2 X_k, X_0'\right)\geq \varepsilon\right\}\right|\right\}_{n=1}^\infty$, we have

$$\frac{1}{n}\left|\left\{k \leq n : d\left(\Delta^2 X_k, X_0'\right) \geq \varepsilon\right\}\right| \nrightarrow 1$$

which contradicts to the fact that $X_0 \neq X_0'$.

Theorem 3.4. Let $\theta = \{k_r\}$ be a lacunary sequence and let $X = \{X_k\}$ a sequence of fuzzy numbers. Then

i) $N_\theta\left(\Delta_p^2\right) \subset S_\theta\left(\Delta^2\right)$,

ii) $m\left(\Delta^2\right) \cap S_\theta\left(\Delta^2\right) \subseteq N_\theta\left(\Delta_p^2\right)$,

iii) If X is Δ^2-bounded then $S_\theta\left(\Delta^2\right) = N_\theta\left(\Delta_p^2\right)$.

Proof. i) If $\varepsilon > 0$ and $X \in N_\theta\left(\Delta_p^2\right)$, we can write

$$h_r^{-1}\sum_{k\in I_r} d\left(\Delta^2 X_k, X_0\right)^p \geq h_r^{-1}\sum_{\substack{k\in I_r\\ d\left(\Delta^2 X_k, X_0\right)\geq\varepsilon}} d\left(\Delta^2 X_k, X_0\right)^p \geq h_r^{-1}\left|\left\{k \in I_r : d\left(\Delta^2 X_k, X_0\right)\geq\varepsilon\right\}\right| \varepsilon^p.$$

Hence X is Δ^2- statistically convergent.

ii) Now suppose that $S_\theta - \lim_k \Delta^2 X_k = X_0$ and $\{X_k\}$ is Δ^2-bounded. Since $X \in m\left(\Delta^2\right)$ there exists a constant $M > 0$ such that $d\left(\Delta^2 X_k, X_0\right) \leq M$ for all k. Let $\varepsilon > 0$ be given and N_ε such that

$$h_r^{-1}\left|\left\{k \in I_r : d\left(\Delta^2 X_k, X_0\right) \geq \left(\frac{\varepsilon}{2}\right)^{1/p}\right\}\right| \leq \frac{\varepsilon}{2M^p}$$

for all $r > N_\varepsilon$ and set $L_r = \left\{k \in I_r : d\left(\Delta^2 X_k, X_0\right) \geq \left(\frac{\varepsilon}{2}\right)^{1/p}\right\}$.

Now for all $r > N_\varepsilon$ we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r} d\left(\Delta^2 X_k, X_0\right)^p &= \frac{1}{h_r}\sum_{k\in L_r} d\left(\Delta^2 X_k, X_0\right)^p + \frac{1}{h_r}\sum_{k\notin L_r} d\left(\Delta^2 X_k, X_0\right)^p \\
&\leq \frac{1}{h_r}\left(\frac{h_r\varepsilon}{2M^p}\right)M^p + \frac{1}{h_r}h_r\frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence $X \to X_0\left(N_\theta\left(\Delta_p^2\right)\right)$.

iii) Follows from (i) and (ii).

Theorem 3.5. $\left|\sigma\left(\Delta^2\right)\right| \subset N_\theta\left(\Delta^2\right)$ if and only if $\liminf_r q_r > 1$.

Proof. (Sufficiency) If $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $1+\delta \leq q_r$ for all $r \geq 1$. Then for $X = (X_k) \in \left|\sigma\left(\Delta^2\right)\right|$, we write

$$\begin{aligned}
\tau_r &= \frac{1}{h_r}\sum_{i=1}^{k_r} d\left(\Delta^2 X_i, X_0\right) - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}} d\left(\Delta^2 X_i, X_0\right) \\
&= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{i=1}^{k_r} d\left(\Delta^2 X_i, X_0\right)\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}} d\left(\Delta^2 X_i, X_0\right)\right).
\end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$. Now $\frac{1}{k_r}\sum_{i=1}^{k_r} d\left(\Delta^2 X_i, X_0\right)$ and $\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}} d\left(\Delta^2 X_i, X_0\right)$ converge to 0. Hence $X \in N_\theta\left(\Delta^2\right)$.

(Necessity) Suppose that $\liminf_r q_r = 1$. Since θ is a lacunary, we can select a subsequence $\left\{k_{r(j)}\right\}$ of θ satisfying $\frac{k_{r(j)}}{k_{r(j)-1}} < 1 + \frac{1}{j}$ and $\frac{k_{r(j)-1}}{k_{r_{(j-1)}}} > j$ where $r_j \geq r_{j-1}+2$. Let A and B denote two distinct fuzzy numbers. Define $X = (X_k)$ such that

$$\Delta^2 X_i = \begin{cases} A, & \text{if } i \in I_{r(j)} \text{ , for some } j = 1, 2, \ldots \\ B, & \text{otherwise.} \end{cases}$$

Then, for any fuzzy numbers T,

$$\frac{1}{h_{r(j)}}\sum_{I_{r(j)}} d\left(\Delta^2 X_i, T\right) = d(A, T)$$

and

$$\frac{1}{h_r}\sum_{I_r} d\left(\Delta^2 X_i, T\right) = d(B, T) \text{ for } r \neq r_j.$$

It follows that $\{X_i\} \notin N_\theta\left(\Delta^2\right)$. However, X is strongly Δ^2-Cesàro summable, since if t is any sufficiently large integer we can find the unique j for which with $k_{(j)-1} < t \leq k_{r(j+1)-1}$ and write

$$\frac{1}{t}\sum_{i=1}^{t} d\left(B, \Delta^2 X_i\right) \leq \frac{k_{r(j)-1} + h_{r(j)}}{k_{r(j)-1}} \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

As $t \to \infty$, it follows that also $j \to \infty$. Hence $\{X_i\} \in \left|\sigma\left(\Delta^2\right)\right|$.

Theorem 3.6. $N_\theta\left(\Delta^2\right) \subset \left|\sigma\left(\Delta^2\right)\right|$ if and only if $\limsup_r q_r < \infty$.

Proof. (Sufficiency) If $\limsup_r q_r < \infty$, there exists $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $X = (X_k) \in N_\theta\left(\Delta^2\right)$ and $\varepsilon > 0$. Then we can find $R > 0$ and $K > 0$ such that $\sup_{i \geq R} \tau_i < \varepsilon$ and $\tau_i < K$ for all $i \in \mathbb{N}$. Suppose that $r > R$ and that t is any integer with $k_{r-1} < t \leq k_r$. Then we can write

$$
\begin{aligned}
\frac{1}{t}\sum_{i=1}^{t} d\left(\Delta^2 X_i, X_0\right) &\leq \frac{1}{k_{r-1}}\sum_{i=1}^{t} d\left(\Delta^2 X_i, X_0\right) \\
&= \frac{1}{k_{r-1}}\left(\sum_{I_1} d\left(\Delta^2 X_i, X_0\right) + \sum_{I_2} d\left(\Delta^2 X_i, X_0\right) + \ldots + \sum_{I_r} d\left(\Delta^2 X_i, X_0\right)\right) \\
&= \frac{k_1}{k_{r-1}}\tau_1 + \frac{k_2 - k_1}{k_{r-1}}\tau_2 + \ldots + \frac{k_R - k_{R-1}}{k_{r-1}}\tau_R + \frac{k_{R+1} - k_R}{k_{r-1}}\tau_{R+1} + \ldots + \frac{k_r - k_{r-1}}{k_{r-1}}\tau_r \\
&\leq \left(\sup_{i \geq 1} \tau_i\right)\frac{k_R}{k_{r-1}} + \left(\sup_{i \geq R} \tau_i\right)\frac{k_r - k_R}{k_{r-1}} < K\frac{k_R}{k_{r-1}} + \varepsilon M.
\end{aligned}
$$

Since $k_{r-1} \to \infty$ as $t \to \infty$, it follows that $\frac{1}{t}\sum_{i=1}^{t} d\left(\Delta^2 X_i, X_0\right) \to 0$. That is, $X = (X_k) \in \left|\sigma\left(\Delta^2\right)\right|$.

(Necessity) Suppose that $\limsup_r q_r = \infty$. In order to prove the result we need to find a sequence $X = (X_k)$ of fuzzy numbers such that $X \in m\left(\Delta^2\right)$, $X \in N_\theta\left(\Delta^2\right)$ and $X \notin \left|\sigma\left(\Delta^2\right)\right|$. Now θ is lacunary, we could construct a subsequence $\left\{k_{r(j)}\right\}$ of θ satisfying $q_{r(j)} > j$. Let A and B be distinct fuzzy numbers. Define $X = (X_k)$ such that

$$
\Delta^2 X_i = \begin{cases} A, & \text{if } k_{r(j)-1} < i \leq 2k_{r(j)-1}\text{, for some } j = 1, 2, \ldots \\ B, & \text{otherwise}. \end{cases}
$$

Then

$$
\tau_{r(j)} = \frac{1}{h_{r(j)}}\sum_{I_{r(j)}} d\left(\Delta^2 X_k, B\right) = d(A, B)\,\frac{k_{r(j)-1}}{k_{r(j)} - k_{r(j)-1}} < \frac{1}{j-1}
$$

and $\tau_r = 0$ if $r \neq r_j$. Hence $\lim_r \frac{1}{h_r}\sum_{I_r} d\left(\Delta^2 X_k, B\right) = 0$. That is $\{X_k\} \in N_\theta\left(\Delta_p^2\right)$.

On the other hand, for the sequence $\{X_k\}$ above and for a fuzzy number T,

$$
\begin{aligned}
\frac{1}{k_{r(j)}}\sum_{i=1}^{k_{r(j)}} d\left(\Delta^2 X_i, T\right) &\geq \frac{1}{k_{r(j)}}\left(\sum_{i=k_{r(j)-1}}^{2k_{r(j)-1}} d(A, T) + \sum_{i=2k_{r(j)-1}}^{2k_{r(j)}} d(B, T)\right) \\
&\geq d(A, T)\,\frac{k_{r(j)-1}}{k_{r(j)}} + d(B, T)\,\frac{k_{r(j)} - 2k_{r(j)-1}}{k_{r(j)}} \\
&\geq d(A, T)\,\frac{k_{r(j)-1}}{k_{r(j)}} + d(B, T)\left(1 - \frac{2}{j}\right) \to d(B, T)
\end{aligned}
$$

and

$$
\frac{1}{2k_{r(j)-1}}\sum_{i=1}^{2k_{r(j)-1}} d\left(\Delta^2 X_i, T\right) \geq \frac{k_{r(j)-1}}{2k_{r(j)-1}} d(B, T) \to \frac{d(B, T)}{2}.
$$

Consequently, for any fuzzy number T, we have

$$\lim_{j\to\infty}\frac{1}{k_{r(j)}}\sum_{i=1}^{k_{r(j)}} d\left(\Delta^2 X_i, T\right) = d(A,T) \neq \frac{d(B,T)}{2} = \lim_{j\to\infty}\frac{1}{2k_{r(j)-1}}\sum_{i=1}^{2k_{r(j)-1}} d\left(\Delta^2 X_i, T\right).$$

Hence $\{X_i\} \notin \left|\sigma\left(\Delta^2\right)\right|$.

The following result is a consequence of Theorem 3.5 and 3.6.

Theorem 3.7. $N_\theta\left(\Delta^2\right) = \left|\sigma\left(\Delta^2\right)\right|$ if and only if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$.

Theorem 3.8. If $X = (X_k) \in N_\theta\left(\Delta^2\right) \cap \left|\sigma\left(\Delta^2\right)\right|$ then $N_\theta - \lim \Delta^2 X_k = |\sigma| - \lim \Delta^2 X_k$.

Proof. Let $|\sigma| - \lim \Delta^2 X_k = X_0$ and $N_\theta - \lim \Delta^2 X_k = X_0'$ and suppose that $X_0 \neq X_0'$. We write

$$\tau_r + \tau_r' = \frac{1}{h_r}\sum_{I_r} d\left(\Delta^2 X_i, X_0\right) + \frac{1}{h_r}\sum_{I_r} d\left(\Delta^2 X_i, X_0'\right) \geq \frac{1}{h_r}\sum_{I_r} d\left(X_0, X_0'\right) = d\left(X_0, X_0'\right)$$

Since $X \in N_\theta\left(\Delta^2\right)$ we have $\tau_r \to 0$. Thus, for sufficiently large r, we have $\delta_r > \frac{1}{2} d\left(X_0, X_0'\right)$. Observe that

$$\begin{aligned}\frac{1}{k_r}\sum_{i=1}^{k_r} d\left(\Delta^2 X_i, X_0\right) &\geq \frac{1}{k_r}\sum_{k\in I_r} d\left(\Delta^2 X_i, X_0\right) = \frac{k_r - k_{r-1}}{k_r}\delta_r \\ &= \left(1 - \frac{1}{q_r}\right)\delta_r > \frac{1}{2}\left(1 - \frac{1}{q_r}\right) d\left(X_0, X_0'\right)\end{aligned}$$

for sufficiently large r. Since $X = (X_i) \in \left|\sigma\left(\Delta^2\right)\right|$, the left hand side converges to 0. So, we must have $q_r \to 1$. That is, $\limsup_r q_r < \infty$. Thus, by Theorem 3.6, $N_\theta\left(\Delta^2\right) \subset \left|\sigma\left(\Delta^2\right)\right|$. Since $N_\theta - \lim_k \Delta^2 X_k = X_0'$, it follows that $|\sigma| - \lim \Delta^2 X_k = X_0'$. Therefore,

$$\frac{1}{t}\sum_{i=1}^{t} d\left(\Delta X_i, X_0'\right) \to \infty.$$

But

$$\frac{1}{t}\sum_{i=1}^{t} d\left(\Delta^2 X_i, X_0'\right) + \frac{1}{t}\sum_{i=1}^{t} d\left(\Delta^2 X_i, X_0\right) \geq d\left(X_0, X_0'\right) > 0$$

which yields a contradiction, since both terms on the left converge to 0.

Theorem 3.9. Let $p = (p_k)$ and $t = (t_k)$ be any two sequences of positive real numbers. Let $0 < p_k \leq t_k$ for each k and $\left(\frac{t_k}{p_k}\right)$ be bounded. Then $N_\theta\left(\Delta_T^2\right) \subset N_\theta\left(\Delta_P^2\right)$,where

$$N_\theta\left(\Delta_P^2\right) = \left\{X = (X_k) : \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left[d\left(\Delta^2 X_k, X_0\right)\right]^{p_k} = 0\right\}.$$

Proof. Let $X \in N_\theta\left(\Delta_T^2\right)$. Write $w_k = \left[d\left(\Delta^2 X_k, X_0\right)\right]^{t_k}$ and $\mu_k = \frac{p_k}{t_k}$, so that $0 < \mu < \mu_k \leq 1$ for each k.

We define the sequences (u_k) and (v_k) as follows:

Let $u_k = w_k$ and $v_k = 0$ if $w_k \geq 1$, and let $u_k = 0$ and $v_k = w_k$ if $w_k < 1$. Then it is clear that for all $k \in \mathbb{N}$, we have $w_k = u_k + v_k$, $w_k^{\mu_k} = u_k^{\mu_k} + v_k^{\mu_k}$. Now it follows that $u_k^{\mu_k} \leq u_k \leq w_k$ and $v_k^{\mu_k} \leq v_k^{\mu}$. Therefore

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r} w_k^{\mu_k} &= \frac{1}{h_r}\sum_{k\in I_r}\left(u_k^{\mu_k} + v_k^{\mu_k}\right) \\
&\leq \frac{1}{h_r}\sum_{k\in I_r} w_k + \frac{1}{h_r}\sum_{k\in I_r} v_k^{\mu}.
\end{aligned}$$

Since $\mu < 1$ so that $\frac{1}{\mu} > 1$, for each r

$$\begin{aligned}
h_r^{-1}\sum_{k\in I_r} v_k^{\mu} &= \sum_{k\in I_r}\left(h_r^{-1}v_k\right)^{\mu}\left(h_r^{-1}\right)^{1-\mu} \\
&\leq \left(\sum_{k\in I_r}\left[\left(h_r^{-1}v_k\right)^{\mu}\right]^{\frac{1}{\mu}}\right)^{\mu}\left(\sum_{k\in I_r}\left[\left(h_r^{-1}\right)^{1-\mu}\right]^{\frac{1}{1-\mu}}\right)^{1-\mu} \\
&= \left(h_r^{-1}\sum_{k\in I_r} v_k\right)^{\mu}
\end{aligned}$$

by Hölder's inequality, and thus

$$h_r^{-1}\sum_{k\in I_r} w_k^{\mu_k} \leq h_r^{-1}\sum_{k\in I_r} w_k + \left(h_r^{-1}\sum_{k\in I_r} v_k\right)^{\mu}.$$

Hence $X \in N_\theta\left(\Delta_P^2\right)$.

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