Fuzzy Strong Pre-semicontinuity

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Abstract: *The theory of fuzzy continuity not only is a significantly basic theory of fuzzy topology and fuzzy analysis but also has wide applications in some other aspects. In this paper, a new class of function is introduced, called fuzzy strongly pre-semicontinuous function. Its properties, its relationship with other functions, examples, and applications are studied.*
Key words: *Fuzzy topology; Pre-semiopen set; Fuzzy strong pre-semicontinuity*

1.Preliminaries

In the paper by (X, δ) or simply by X we mean a fuzzy topological space in the Chang's[4] sense, briefly fts. A^o, A^-, A_o, A_- and A' denote the interior, closure, semiinterior, semiclosure and complement of fuzzy set A, respectively. A fuzzy set A in X is called pre-semiopen iff $A \leqslant (A^-)_o$, and pre-semiclosed iff $A \geqslant (A^o)_-$[1]. $PSO(X)$ and $PSC(X)$ denote the family of pre-semiopen sets and family of pre-semiclosed sets of an fts X, respectively. The $A_\Delta = \cup\{B: B \in PSO(X), B \leqslant A\}$ and $A_\sim = \cap\{B: B \in PSC(X), A \leqslant B\}$ are called the pre-semiinterior and pre-semiclosure of fuzzy set A[1], respectively. Since the union (intersection) of any two fuzzy pre-semiclosed (pre-semiopen) sets need not be a pre-semiclosed (pre-semiopen) set [1], $A \in PSC(X)$ and $B \in PSC(X)$ do not necessarily lead to $A \cup B \in PSC(X)$. Let

$UPSC(X)=\{A \in PSC(X):$ for each $B \in PSC(X), A \cup B \in PSC(X)\}$,

$IPSO(X)=\{A \in PSO(X):$ for each $B \in PSO(X), A \cap B \in PSO(X)\}$.

Clearly, $\delta \subset IPSO(X) \subset PSO(X)$.

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2. Fuzzy Strongly Pre-semicontinuous Functions

Definition 2.1. A function $f:(X, \delta)\to(Y, \tau)$ is said to be fuzzy strongly pre-semicontinuous if $f^{-1}(B)\in IPSO(X)$ for each $B\in\tau$.

Corollary 2.2. A function $f:(X, \delta)\to(Y, \tau)$ is fuzzy strongly pre-semicontinuous iff $f^{-1}(B)\in UPSC(X)$ for each $B'\in\tau$.

Theorem 2.3. Let $f:(X, \delta)\to(Y, \tau)$ be a fuzzy strongly pre-semicontinuous function. Then:

(1) $f(A_{\sim})\leqslant(f(A))^{-}$ for each fuzzy set A in X.

(2) $(f^{1}(B))_{\sim}\leqslant f^{1}(B^{-})$ for each fuzzy set B in Y.

(3) $f^{1}(B^{o})\leqslant(f^{1}(B))_{\triangle}$ for each fuzzy set B in Y.

(4) There is a base β for τ such that $f^{-1}(B)\in IPSO(X)$ for each $B\in\beta$.

(5) For each fuzzy point x_{a} in X and each $B\in\tau$ with $f(x_{a})\in B$, there exists an $A\in IPSO(X)$ such that $x_{a}\in A$ and $f(A)\leqslant B$.

Proof. (1): Let A be a fuzzy set in X. Then $(f(A))^{-}$ is a fuzzy closed set in Y. Since f is fuzzy strongly pre-semicontinuous, $f^{1}((f(A))^{-})\in UPSC(X)$, and

$$A_{\sim}\leqslant(f^{-1}f(A))_{\sim}\leqslant(f^{-1}((f(A))^{-}))_{\sim}=f^{-1}((f(A))^{-}).$$

Thus, $f(A_{\sim})\leqslant ff^{-1}((f(A))^{-})\leqslant(f(A))^{-}$.

(2): Let B be a fuzzy set in Y. By (1), $f((f^{-1}(B))_{\sim})\leqslant(ff^{-1}(B))^{-}\leqslant B^{-}$. Thus, $(f^{-1}(B))_{\sim}\leqslant f^{-1}f((f^{-1}(B))_{\sim})\leqslant f^{-1}(B^{-})$.

(3): Let B be a fuzzy set in Y. By (2), $f^{-1}(B^{r})\geqslant(f^{-1}(B'))_{\sim}=((f^{-1}(B))')_{\sim}$. Thus,

$$f^{-1}(B^{o})=f^{1}(B^{r\prime})=(f^{-1}(B^{r}))'\leqslant(((f^{-1}(B))')_{\sim})'=(f^{-1}(B))_{\triangle}.$$

(4): Obvious.

(5): Let f be fuzzy strongly pre-semicontinuous, x_{a} be a fuzzy point in X and $B\in\tau$ such that $f(x_{a})\in B$. Then $x_{a}\in f^{-1}(B)$. Let $A=f^{-1}(B)$, then $A\in IPSO(X)$. We have $f(A)=ff^{-1}(B)\leqslant B$.

Theorem 2.4. Let $f:(X, \delta)\to(Y, \tau)$ be a fuzzy strongly pre-semicontinuous function, and one-to-one and onto. Then $(f(A))^{o}\leqslant f(A_{\triangle})$ for each fuzzy set A in X.

Proof. Let f be fuzzy strongly pre-semicontinuous and A be any fuzzy set in X. Then $f^{-1}((f(A))^{o})\in UPSO(X)$. By Theorem 2.3 and the fact that f is one-to-one, we have $f^{-1}((f(A))^{o})\leqslant(f^{-1}f(A))_{\triangle})=A_{\triangle}$. Again, since f is onto, we have

$$(f(A))^{o}=ff^{-1}((f(A))^{o})\leqslant f(A_{\triangle}).$$

Proposition 2.5. If $f: X \to Y$ is a fuzzy strongly pre-semicontinuous function and $g: Y \to Z$ is a fuzzy continuous function, then gf is fuzzy strongly pre-semicontinuous.

Theorem 2.6. Let $f: X_1 \to X_2$ and $g: X_3 \to X_4$ be fuzzy strongly pre-semicontinuous. Then the product $f \times g: X_1 \times X_3 \to X_2 \times X_4$ is fuzzy pre-semicontinuous.

Proof. Let $B = \cup(A_i \times B_j)$, where the A_i's and B_j's are open sets of X_2 and X_4, respectively. B is a open set of $X_2 \times X_4$. Then

$$\begin{aligned}(f\times g)^{-1}(B) &= (f\times g)^{-1}(\cup(A_i\times B_j)) \\ &= \cup(f\times g)^{-1}(A_i\times B_j) \\ &= \cup(f^{-1}(A_i)\times g^{-1}(B_j)).\end{aligned}$$

That $(f\times g)^{-1}(B)$ is a pre-semiopen set follows from Theorem 1.7 and 1.6 in [1]. Thus, $f\times g$ is fuzzy pre-semicontinuous.

Theorem 2.7. Let $p_i: X_1 \times X_2 \to X_i$ $(i=1,2)$ be the projection of $X_1 \times X_2$ on X_i. If f: X $\to$ $X_1 \times X_2$ is fuzzy strongly pre-semicontinuous, then $p_i f$ is also fuzzy strongly pre-semicontinuous.

Proof. This follows directly from Proposition 2.5.

Theorem 2.8. Let $f: X_1 \to X_2$ be a function. If the graph $g: X_1 \to X_1 \times X_2$ of f is fuzzy strongly pre-semicontinuous, then f is also fuzzy strongly pre-semicontinuous.

Proof. This follows directly from Theorem 2.7.

3. Examples

Definition 3.1[1]. A function $f: (X, \delta) \to (Y, \tau)$ is said to be fuzzy pre-semicontinuous if $f^{-1}(B) \in PSO(X)$ for each $B \in \tau$.

Clearly, the following statements are valid:

fuzzy continuity $\Rightarrow$ strong pre-semicontinuity $\Rightarrow$pre-semicontinuity

None of the converses need to be true. We give the following examples.

Example 3.2. Let $X=[0,1]$ and A, B, C be fuzzy sets in X defined as follows:

$A(x)=0.1, x\in[0,1]$; $B(x)=0.5, x\in[0,1]$; $C(x)=0.4, x\in[0,1]$.

Then $\delta=\{0, A, B, 1\}$ and $\tau=\{0, C, 1\}$ are fuzzy topologies on X. Let $f:(X, \delta) \rightarrow (X, \tau)$ be an identity mapping. Clearly, f is not fuzzy continuous; and f is fuzzy strongly pre-semicontinuous.

Example 3.3. Let $X=\{x, y, z\}$ and A, B, C be fuzzy sets in X defined as follows:

A(x)=0.2, A(y)=0.4, A(z)=0.5;
B(x)=0.8, B(y)=0.8, B(z)=0.6;
C(x)=0.3, C(y)=0.2, C(z)=0.4.

Then $\delta=\{0, A, B, 1\}$ and $\tau=\{0, C, 1\}$ are fuzzy topologies on X. Let f: $(X, \delta)\rightarrow(X, \tau)$ be an identity mapping. In (X, δ), by easy computations it follows that $C\leqslant(C^{-})_{o}=(A')_{o}=A'$, i.e. $f^{-1}(C)=C$ is a pre-semiopen set. Hence, f is fuzzy pre-semicontinuity. Because $A\cap C=B'$ and $B'\nleqslant(B^{r})_{o}=(B')_{o}=0$, $A\cap C$ is not a pre-semiopen set in (X, δ), i.e. $f^{-1}(C)=C\notin IPSO(X, \delta)$. Thus, f is not fuzzy strongly pre-semicontinuous.

4. Applications

Definition 4.1[2]. A fuzzy set A is called a PS-connected set if A cannot be represented as a union of two PS-separated non-null sets.

Theorem 4.2. Every fuzzy strongly pre-semicontinuous image of a fuzzy PS-connected set is fuzzy connected.

Proof. Let f: $X\rightarrow Y$ be a fuzzy strongly pre-semicontinuous function and A be a fuzzy PS-connected set in X. If possible, let $f(A)$ be not fuzzy connected in Y. Then there exist two separated non-null sets B and C in Y such that $f(A)=B\cup C$. Put $E=A\cap f^{-1}(B)$ and $F=A\cap f^{-1}(C)$. Then

$$E\cup F=A\cap(f^{-1}(B)\cup f^{-1}(C))=A\cap(f^{-1}f(A))=A,$$

and

$$E_{\sim}\cap F=(A\cap f^{-1}(B))_{\sim}\cap(A\cap f^{-1}(C))\leqslant A_{\sim}\cap(f^{-1}(B))_{\sim}\cap A\cap f^{-1}(C)$$
$$\leqslant A\cap f^{-1}(B^{-})\cap f^{-1}(C)=A\cap f^{-1}(B^{-}\cap C)=A\cap f^{-1}(0_{Y})=0_{X}.$$

Analogously, $E\cap F_{\sim}=0_X$. Again $E\neq 0_X$, in fact, if $E=0_X$, then $A=F=A\cap f^{-1}\}(C)$. And so $A\leqslant f^{-1}(C)$, and $f(A)\leqslant F$. Hence, $B\leqslant C$, This is a contradiction. Analogously, $F\neq 0_X$. Thus, A is not fuzzy PS-connected in X.

Definition 4.3. A fuzzy topological space (X, δ) is called fuzzy IPSO-compact (countably IPSO-compact) if for every cover (countable cover) $\{V_{\alpha}: V_{\alpha}\in IPSO(X)\}$ of X, there exists a finite subcover of X.

Theorem 4.4. Every surjection fuzzy strongly pre-semicontinuous image of a fuzzy IPSO-compact space is fuzzy compact.

Proof. Let $f: X \to Y$ be a surjection fuzzy strongly pre-semicontinuous function of a fuzzy IPSO-compact space X to a fuzzy topological space Y. Let $\{V_a: a \in J\}$ be a fuzzy open cover of Y. Then $f^{-1}(V_a) \in IPSO(X)$ for each $a \in J$, and $\varphi=\{f^{-1}(V_a): a \in J\}$ is a cover of X. Since X is fuzzy IPSO-compact, there exists a finite subset J_0 of J such that $\cup\{f^{-1}(V_a): a \in J_0\}=I_X$. Now

$$I_Y=f(I_X)=f(\cup\{f^{-1}(V_a): a \in J_0\})=\cup\{ff^{-1}(V_a): a \in J_0\}$$
$$\leqslant \cup\{V_a: a \in J_0\}.$$

Therefore, Y is fuzzy compact.

Corollary 4.5. Every surjection fuzzy strongly pre-semicontinuous image of a countably IPSO-compact space is countably compact.

References

[1]. S.Z.Bai, Fuzzy pre-semiopen sets and fuzzy pre-semicontinuity, Proc.ICIS'92 (1992)918-920.

[2]. S.Z.Bai, PS-connectedness of L-fuzzy sets, in press.

[3]. S.Z.Bai, L-fuzzy PS-compactness, IJUFKS, 10(2002)201-209.

[4]. C.L.Chang, Fuzzy topological spaces, J.Math.Anal.Appl.24(1968) 182-190.

[5]. Y.M.Liu, M.K.Luo, Fuzzy Topology, World Sci. Publishing, Singapore, 1988.

[6]. G.J.Wang, Theory of L-fuzzy Topological Spaces, Press of Shaanxi Normal University, Xian, 1988.