WAVELET DENOISING BASED ON SYMMETRIC WAVELETS AND THE IDEA OF NEYMAN AND PEARSON

Zhen Yan, Shao Fang-ming and Zhao Lian-chang

(*Dept. of Mathematics and Physics*, Dalian Maritime University, Dalian, Liaoning 116026, China)

Abstract-In this paper, we first develop a new method to design symmetric wavelets, and then introduce the idea of Neyman and Pearson to give a wavelet denoising method. The simulation results show the good performance of our proposed method.

Keywords-Linear phase, wavelet transform, denoising

I. INTRODUCTION

Wavelet thresholding technique[9] for denoising is viewed as one of the breakthroughs in wavelet theory and its applications. In this paper, we try a novel denoising method based on symmetric wavelets and the idea of Neyman and Pearson. Our method consists of two steps. Firstly, we introduce the theory of wavelets with convolution-type orthogonality conditions[8], which uses free parameters to develop a new way to construct biorthogonal wavelets. Based on this theory, we find an easier and more efficient method to construct symmetric wavelets. Such wavelets contain the symmetric wavelets obtained in [8]. And then we apply a novel wavelet shrinkage/thresholding technique based on the idea of Neyman and Pearson[10] to give a wavelet denoising method. We do experiments using this method and new constructed wavelets. The improving mean square error of signal denoising makes it possible for this method for further use.

II. SYMMETRIC WAVELETS

Multiresolution analysis[4] is a very important concept to construct wavelet functions. Let $\phi(x)$ be a scaling function in $L^2(R)$ satisfying a two-scale relation

$$\phi(x)=\sum_{n\in Z}\alpha_n\sqrt{2}\phi(2x-n). \tag{1}$$

Where Z denotes the set of integers, and α_n is real parameter. Define spaces

$$V_0=\{\sum_k c_{0,k}\phi(\cdot-k),c_{0,k}:real,\sum_k c_{0,k}^2<+\infty\}.$$
$$V_1=\{\sum_k c_{1,k}\sqrt{2}\phi(2\cdot-k),c_{1,k}:real,\sum_k c_{1,k}^2<+\infty\} \tag{2}$$

Obviously, V_0 is a subspace of V_1 and has an orthogonal complement W_0 in V_1. The orthonormal basis of W_0 in the form of $\{\psi(\cdot-k),k\in Z\}$ is called a wavelet function. In order to construct wavelet functions with linear phases, [8] introduced a function $\phi_p(x)$ in $L^2(R)$ and let it satisfy a two-scale relation

$$\phi_p(x)=\sum_n(p*\alpha)_n\sqrt{2}\phi_p(2x-n). \tag{3}$$

Here, $P=(p_m)_{m=-M,\cdots,M}$ are real parameters, the symbol * denotes the convolution

$$(p*\alpha)_n=\sum_{|n-l|\le M}p_{n-l}\alpha_l. \tag{4}$$

It has been proved that under some conditions of parameters (p_m), the orthonormality conditions imposing on the scaling and wavelet functions become

$$\sum_l \left(\sum_{m=-M}^{M} p_m \alpha_{l+2k-m} \right) \alpha_l = \delta_{k,0}, \quad k \in Z \tag{5}$$

$$(\phi(\cdot - k), \psi(\cdot))_p = \int \phi_p(x-k)\psi(x)dx = 0. \tag{6}$$

And let $\alpha(\varsigma) = \sum_{m=-M}^{M} p_m e^{-im\varsigma}$ and $\psi(x) = \sum_n \beta_n \sqrt{2}\phi(2x-n)$, the wavelet coefficient β_n can be given as

$$\beta_n = \sum_l (-1)^{l-1} \frac{\alpha_l}{\pi} \int_0^{\pi} \sqrt{a(\varsigma+\pi)/a(\varsigma)} \cos(n+l+1)\varsigma d\varsigma. \tag{7}$$

At present, (5) has not been solved analytically in the general case. Here we try to use (5) to obtain symmetric wavelets in the general case.

First we add some limiting conditions to the scaling coefficients, that is

$$\alpha_n = \alpha_{-n}, \quad (\text{n}=-N,\cdots,N). \tag{8}$$

An additional requirement for α_n is

$$\sum_n \alpha_n = \sqrt{2}. \tag{9}$$

This is derived by integrating both sides of the two-scale relation (1). From the condition of wavelet $\int \psi(x)dx = 0$, it follows that

$$\sum_n (-1)^n \alpha_n = 0. \tag{10}$$

The conditions that parameters (p_m) have to satisfy have been discussed in [8], and here we consider it true that $p_m = p_{-m}$ $(m = 0,\cdots,M)$.

Now, from these regularity conditions and equation (5) we can construct symmetric scaling coefficients for different *N* and *M*. Then from equation (7) the wavelet coefficients can be obtained. Since $\alpha_n = \alpha_{-n}$, we have $\beta_{n-2} = \beta_{-n}$. It means wavelet function $\psi(x)$ is symmetric with respect to x=-1/2. We take example for *N*=2 and *M*=3 to see the constructing process.

When *N*=2 and *M*=3, (5) can be written as

$$\sum_{l=-2}^{2} \left(\sum_{m=-3}^{3} p_m \alpha_{l+2k-m} \right) \alpha_l = \delta_{k,0}. \tag{11}$$

Equations (11) for k = 1, 2 coincide with those for k = -1, -2, respectively. So it suffices to consider (11) for k = -2, -1, 0, that is

$$2p_3(a_1a_2 + a_0a_1) + p_2(2a_0a_2 + a_1^2) + 2p_1a_1a_2 + p_0a_2^2 = 0 \tag{12}$$

$$2p_3(a_1a_2 + a_0a_1) + p_2(a_0^2 + 2a_1^2 + 3a_2^2) + p_1(2a_0a_1 + 4a_1a_2) + p_0(a_1^2 + 2a_0a_2) = 0 \tag{13}$$

$$4p_3a_1a_2 + p_2(4a_0a_2 + 2a_1^2) + 4p_1(a_1a_2 + a_0a_1) + p_0(a_0^2 + 2a_1^2 + 2a_2^2) = 1 \tag{14}$$

Additional requirements for α_n are

$$a_0 + 2a_1 + 2a_2 = \sqrt{2} \tag{15}$$

$$a_0 - 2a_1 + 2a_2 = 0 \tag{16}$$

We assume another condition for p_m like [8]

$$p_0 + 2p_1 + 2p_2 + 2p_3 = 1. \tag{17}$$

Let $r_{n+l+1} = \frac{1}{\pi} \int_0^{\pi} \sqrt{a(\varsigma+\pi)/a(\varsigma)} \cos(n+l+1)\varsigma d\varsigma$ then β_n can be computed as follow

$$\beta_n = \sum_{l=-2}^{2} (-1)^{l-1} \alpha_l r_{n+l+1} = -\alpha_2 (r_{n-1} + r_{n+3}) + \alpha_1 (r_n + r_{n+2}) - \alpha_0 r_{n+1} \qquad (18)$$

TABLE I SOME GROUPS OF SCALING AND WAVELET COEFFICIENTS

i: N=4, M=1, p_0=2, $p_1 = p_{-1}$=-0.5
α_n : -0.04419417382416, -0.08838834764832, 0.35355339059327, 0.44194173824159, 0.08838834764832, 0.44194173824159, 0.35355339059327, -0.08838834764832, -0.04419417382416
β_n : 0.00001746714644, 0.00005485676480, -0.00002778010695, -0.00000316953318, 0.00045291636238, 0.00616971919851, 0.01390427372616, -0.16407233201906, -0.27926340158108, 0.36190206932572, 0.12176748836674, 0.36190206932572, -0.27926340158108, -0.16407233201906, 0.01390427372616, 0.00616971919851, 0.00045291636238, -0.00000316953318, -0.00002778010695, 0.00005485676480, 0.00001746714644
ii: N=5, M=1, p_0= 1.16592899658748, $p_1 = p_{-1}$= -0.08296449829374
α_n : 0.00005406483306, 0.00037989596670, -0.01040729656104, -0.05475153755612, 0.36390662232125, 0.81585006436538, 0.36390662232125, -0.05475153755612, -0.01040729656104, 0.00037989596670, 0.00005406483306
β_n : 0.00000793387534, 0.00001898330148, -0.00006925289659, 0.00004408223181, 0.00000234673982, 0.00000790782076, -0.00092814269799, -0.00577406870711, 0.07834448658488, 0.31214310351279, -0.76759707408512, 0.31214310351279, 0.07834448658488, -0.00577406870711, -0.00092814269799, 0.00000790782076, 0.00000234673982, 0.00004408223181, -0.00006925289586, 0.00001898330148, 0.00000793387534
iii: N=2, M=3, p_0=13, $p_1 = p_{-1}$ =-8.1875, $p_2 = p_{-2}$ =2.5, $p_3 = p_{-3}$=-0.31250
α_n : 0.08838834764832, 0.35355339059327, 0.53033008588991, 0.35355339059327, 0.08838834764832
β_n : -0.00028435018481, -0.00135720785571, -0.00440912154448, -0.00970692956834, 0.00297164939604, 0.05977127583183, 0.06156259401475, -0.21709369497684, 0.06156259401475, 0.05977127583183, 0.00297164939604, -0.00970692956834, -0.00440912154448, -0.00135720785571, -0.00028435018481

Note: Let α_n and β_n be zero-padded to equal length, and $\{\alpha_n\}$ is the reconstruction low-pass filter, $\{(-1)^n \alpha_n\}$ is the decomposing high-pass filter. Regulate the sum of the vector $\{\beta_n^1\} := \{(-1)^{n+1} \beta_n\}$ to $\sqrt{2}$, then $\{\beta_n^1\}$ is used as the decomposing low-pass filter and $\{(-1)^{n+1} \beta_n^1\}$ is the reconstruction high-pass filter.

III. WAVELET DENOISING

Suppose the signal $X = [x_0, x_1, x_2, \cdots, x_{N-1}]$ has been corrupted by white noise, that is

$$x_i = s_i + n_i \qquad i = 0, 1, 2, \cdots, N\text{-}1.$$

s_i is the true value of the signal and n_i is the noise value at the moment i. Let $\hat{S}$ is the estimate of the signal. The mean square error is computed as follow

$$\xi(\hat{S}, S) = \frac{1}{N} \left\| \hat{S} - S \right\|^2 = \frac{1}{N} \sum_{i=0}^{N-1} (\hat{s}_i - s_i)^2 \qquad (19)$$

Donoho and Johnstone[9] have demonstrated that wavelet denoising is a powerful tool for removing the noisy component of a corrupted data sequence. If the source signal is smooth enough, this thresholding technique will perform well since the source signal will only contribute to a few wavelet coefficients. However, for source signals that are not smooth at some spatial points, such as radar and ultrasound signals, special attention is still required for those wavelet coefficients associated with singularity points because they carry important information about the transmitted signal. In our study, we introduce the thresholding technique based on the idea of Neyman and Pearson[10] to resolve the problem. First, the following binary test is applied

$$H_0 : cd_{jk} \sim N(0,\sigma) \quad \text{versus} \quad H_1 : cd_{jk} \sim N(cd^n_{jk},\sigma) \qquad j=1,2,\cdots,J.$$

If $P\{cd_{jk} | H_1\} / P\{cd_{jk} | H_0\} > P\{H_0\} / P\{H_1\}, H_1$ is assured; otherwise, H_0 is taken. The confidence interval is chosen as $[-\lambda, \lambda]$ with the confidence level $\alpha = P(|cd_{jk}| \le \lambda)$. It is known that for any given $\alpha \in (0,1)$ the corresponding is optimized in the Neyman and Pearson sense. Let $\alpha \in (0,1)$ and $\lambda = \sqrt{2}\sigma erfinv(\alpha)$ [where erfinv(*) is the inverse function of $erf(y) = 2/(\sqrt{\pi})\int_0^y \exp(-t^2)dt$], and define threshold/shrinkage as follow

Threshold:

$$\tilde{cd}_{jk} = \begin{cases} cd_{jk}, if |cd_{jk}| \ge \lambda \\ 0, otherwise. \end{cases}$$

This threshold has a priori information of noise distribution that is used to define the shrinkage instead of choosing a constant threshold $\sigma\sqrt{2\log n}$. The wavelet denoising method based on the idea of Neyman and Pearson can be summarized as follows

(a) Wavelet transform of the source signal and index the wavelet coefficients cd_{jk} (j= 1, 2,$\cdots$, J), J means the level of decomposing and the scaling coefficients remain still.

(b) Apply the novel threshold to test each wavelet coefficient and obtain the noise free wavelet coefficients $\tilde{cd}_{jk}$.

(c) Inverse wavelet transform to recover source signal.

IV. SIMULATION RESULTS

We did experiments on the four typical signals, Blocks, Bumps, Heavy Sine and Doppler in MATLAB. The length of the signal sequences was 2048, signal-to-noise ratio was 7, initial number was 2055615866, the level of wavelet decomposing was 5 with global shrinkage, and the standard deviation of signal σ was chosen as 1. Comparisons were made with the denoising method of applying the threshold based on the idea of Neyman and Pearson with the confidence level $\alpha = 0.9995$ and the second group of wavelet in TABLE I[denoted by *New*], and of Donoho's hard thresholding technique with wavelets 'bior3.9' and 'db10'[denoted by *bior3.9* and *db10*]. The mean square errors were given respectively in TABLE II, which showed the good performance of our proposed method.

TABLE II MSE of Signal Denoising

MSE	Blocks	Bumps	HesviSine	Dopple
Original signal	1.0418	1.0418	1.0418	1.0418
bior3.9	0.2660	0.2004	0.1571	0.2024
db10	0.2814	0.1302	0.0332	0.1011
New	0.1410	0.1243	0.0343	0.1167

V.CONCLUTIONS

This paper considers the wavelet-denoising problem, and a signal denoising method based on the idea of Neyman and Pearson and new symmetric wavelets is proposed. Simulation results show its good performance. The proposed method improves the mean square error of signal denoising. However, further research need to be done on the influence of the parameters on the quality of the wavelets.

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