Some properties of the direct product of fuzzy algebras

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Abstract: In this paper first we give some properties of the direct product of two fuzzy subsets and fuzzy algebras. Then we study the product structure of the fuzzy factor algebras.

Keywords: fuzzy algebra; fuzzy ideal; fuzzy factor algebra ; direct product of fuzzy algebra; algebra homomorphism

1. Introduction

Wenxiang Gu and LuTu[3] introduced the notion of fuzzy algebras over fuzzy fields and obtained some fundamental results pertaining to this notion. In[3] the fuzzy quotient algebras are studied, but the fuzzy quotient algebras of [3] are not fuzzy sets. Thus paper [1] gave the definition of fuzzy factor algebra. Based on paper[1],this paper offers some properties of the direct product of two fuzzy subsets and fuzzy algebras. And then, the product structure of the fuzzy factor algebras is studied.

2. Preliminaries

Here we recall some definitions and properties.

Definition 2.1[5]. Let A and B be fuzzy subsets of the sets G and H,respectively. The direct product of A and B,denoted by $A \times B$, is the function defined by setting for all x in G and y in H,

$$(A \times B)(x, y) = \min(A(x), B(y)).$$

Definition 2.3[6]. Let A be a fuzzy subset of a set G and let $t \in [0,1]$. The $A[t] = \{x \in G / A(x) \geq t\}$ is called a level subset of A.

Definition 2.3[3]. Let X be a field and F a fuzzy set of X. If the following conditions hold:

(1) $F(x+y) \geq F(x) \wedge F(y), x, y \in X;$

(2) $F(-x) \geq F(x), x \in X;$

(3) $F(xy) \geq F(x) \wedge F(y), x, y \in X;$

(4) $F(x^{-1}) \geq F(x), x(\neq 0) \in X$.

We call F a fuzzy field of X, denoted by (F,X). Also, (F,X) *is* called a fuzzy field of X.

Definition 2.4[3]. Let (F,X) be a fuzzy field of the field X, Y a algebra over X and A a fuzzy set of Y. Suppose the following conditions holds:

(1) $A(x+y) \geq A(x) \wedge A(y), x, y \in Y$;

(2) $A(\lambda x) \geq F(\lambda) \wedge A(x), \lambda \in X$ and $y \in Y$;

(3) $A(xy) \geq A(x) \wedge A(y), x, y \in Y$;

(4) $F(1) \geq A(x), x \in Y$.

Then we call (A,Y) a fuzzy algebra over fuzzy field (F,X). Also, A is called a fuzzy algebra of Y.

Definition 2.4[4]. Let (A,Y) be a fuzzy algebra over fuzzy field (F,X), a fuzzy algebra (A,Y) will be called a fuzzy ideal, if $A(xy) \geq A(x) \vee A(y)$ for all x,y in Y.

Let Y be algebra over the field X, B be a fuzzy ideal of Y. Dang in[4] proved that Y/B was a algebra. Let A be a fuzzy algebra of Y, then A/B is a fuzzy set on Y/B defined as follows:

$A/B : Y/B \rightarrow [0,1]$ satisfying

$A/B(a+B) = \sup\limits_{x+B=a+B} A(x)$ for any a+B in Y/B.

Paper[1] prove that A/B is a fuzzy algebra of Y/B, and call it the fuzzy factor algebra of A with respect to B.

3. Some properties of the direct product of fuzzy algebras

G and G' will denote general algebras over fields X in this paper.

Proposition 3.1. If A and B are fuzzy algebras(ideals) of the algebras G and G', respectively, then $A \times B$ is a fuzzy algebra(ideal) of $G \times G'$.

Proposition 3.1. Let A,B be separately fuzzy algebras of G, G' . For any $\alpha \in [0,1], (A \times B)[\alpha] = A[\alpha] \times B[\alpha]$ holds.

Proposition 3.2. Let A and B be fuzzy subsets of the algebras G and G'. For any $\alpha \in [0,1], (A \times B) = A[\alpha] \times B[\alpha]$ holds.

Proposition 3.3. Let A and B be fuzzy subsets of the algebras G and G', respectively. Suppose that o and o' are the zero elements of G and G', respectively. If $A \times B$ is a fuzzy algebra of $G \times G'$, then at least one of the following two statements must hold.

(1) $B(o') \geq A(x)$ for all x in G;

(2) $A(o) \geq B(y)$ for all y in G'.

Proposition 3.4. Let A and B be fuzzy subsets of the algebras G and G', respectively, such that $A(x) \leq B(o')$ for all x in G, o' being the zero element of G'. If $A \times B$ is a fuzzy algebra of $G \times G'$, then A is a fuzzy algebra of G.

Proposition 3.5. let A and B be fuzzy subsets of the algebras G and G', respectively, such that $B(x) \leq A(o)$ for all $x \in G', o$ being the zero element of G. If $A \times B$ is a fuzzy algebra of $G \times G'$. then B is a fuzzy algebra of G'.

From proposition 3.2,3,3 and 3.4 we have the following corollary.

Corollary 3.6. Let A and B be fuzzy subsets of the algebras G and G', respectively. If $A \times B$ is a fuzzy algebra of $G \times G'$, then either A is a fuzzy algebra of G or B is a algebra of G'.

4. The structure of product fuzzy factor algebras

Proposition 4.1. Let A be a fuzzy algebra of G and B a fuzzy ideal of G. If A has the sup property, then $(A/B)[\alpha] = A[\alpha]/B$ for any $\alpha \in [0,1]$.

Proposition 4.2. Let A,B be fuzzy algebras of G and G' respectively. If A has sup property and there is a mapping $f: G \to G'$ such that $f(A[\alpha]) = B[\alpha]$ for any $\alpha \in [0,1]$, then $f(A) = B.$

Proposition 4.3. Suppose A and B are fuzzy algebras of G and G', respectively, with the sup property. If A', B' are separately fuzzy ideals of G and G', then $A/A', A \times B$ has the sup property.

Definition 4.1. Let f be homomorphism(isomorphism) from the algebra G onto the algebra G'.A and A' are separately the fuzzy algebra G and G'.If $A' = f(A)$, then f is said to be homomorphism(isomorphism) from A onto A'.

Proposition 4.4. Let A,B are fuzzy algebras of G and G' respectively, with the sup property. Suppose A', B' are separately fuzzy ideals of G and G', $A'(o) = B'(o')$. Then $A \times B / A' \times B'$ is isomorphic to $A/A' \times B/B'$.

Proof. By proposition 3.1, we know $A \times B$ is a algebra of $G \times G'$,and $A' \times B'$ is a fuzzy ideal of $G \times G'$.

Now we come to prove that $A \times B / A' \times B'$ is isomorphic to $A/A' \times B/B'$ .For any $\alpha \in [0,1]$,it is clear that $A[\alpha] \times B[\alpha]/A' \times B'$ is empty iff $A[\alpha]/A' \times B[\alpha]/B'$ is empty. Suppose both of them are nonempty. Let

$$f_\alpha : A[\alpha] \times B[\alpha]/A' \times B' \to A[\alpha]/A' \times B[\alpha]/B'$$

satisfy

$$f_\alpha((a,b)+A'\times B')=(a+A',b+B'),a\in A[\alpha],b\in B[\alpha].$$

If $(a,b)+A'\times B'=(c,d)+A'\times B'$, then

$$A'\times B'((c,d)-(a,b))=A'\times B'(o,o')$$
$$A'\times B'(c-a,d-b)=A'\times B'(o,o')$$
$$A'(c-a)\wedge B'(d-b)=A'(o)\wedge B'(o)$$

Since $A'(o)=B'(o)$ we have

$$A'(c-a)=A'(o),B'(d-b)=B'(o'),$$

So $a+A'=c+A',b+B'=d+B'$. Hence $(a+A',b+B')=(c+A',d+B')$ holds.

Thus f_α is an one-valued mapping.

It is clear that if $(a+A',b+B')=(c+A',d+B')$ holds then $(a,b)+A'\times B'=(c,d)+A'\times B'$ holds. So that f_α is a monomorphism..

For any $(a,b)+A'\times B',(c,d)+A'\times B'\in G\times G'/A'\times B',\lambda\in X$, we have

$$f_0((a,b)+A'\times B'+(c,d)+A'\times B')$$
$$=f_0((a+c,b+d)+A'\times B')$$
$$=((a+c)+A',(b+d)+B')$$
$$=(a+A',b+B')+(c+A',d+B')$$
$$=f_0((a+b)+A'\times B')+f_0((c+d)+A'\times B')$$
$$f_0((a,b)+A'\times B')((c,d)+A'\times B')$$
$$=f_0((ac,bd)+A'\times B')$$
$$=(ac+A',bd+B')$$
$$=(a+A',b+B')(c+A',d+B')$$
$$=f_0((a,b)+A'\times B')f_0((c,d)+A'\times B').$$
$$f_0(\lambda((a,b)+A'\times B'))$$
$$=f_0((\lambda a,\lambda b)+A'\times B')$$
$$=(\lambda a+A',\lambda b+B')$$
$$=\lambda(a+A',b+B')$$
$$=\lambda f_0((a,b)+A'\times B')$$

Hence f_0 is an algebra isomorphism.

For any $\alpha\in[0,1]$,

$$
\begin{aligned}
& f_0((A\times B/A'\times B')[\alpha]) \\
& = f_0((A\times B)[\alpha]/A'\times B') \\
& = f_0((A[\alpha]\times B[\alpha])/A'\times B') \\
& = f_\alpha((A[\alpha]\times B[\alpha])/A'\times B') \\
& = A[\alpha]/A'\times B[\alpha]/B' \\
& = (A/A')[\alpha]\times(B/B')[\alpha] \\
& = (A/A'\times B/B')[\alpha]
\end{aligned}
$$

Hence $A\times B/A'\times B'$ is isomorphic to $A/A'\times B/B'$.

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