On a-order-convexity of fuzzy syntopogenous spaces

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Abstract:

In this paper, we combine L-fuzzy syntopogenous structure on X with algebraic structure on X. First, the *-increasing and *-decreasing spaces have been studied. Second, we define a-order-convexity on syntopogenous structures $(X,S,\leqslant)$. Third, the equivalent description of a-order-convexity have been given. Finally, some important properties of a-order-convexity have been obtained.

Keywords:

Fuzzy topology; Order; Algebra; convexity.

1 Preliminaries

In this paper, $L=\langle L,\wedge,\vee,'\rangle$ always denotes a completely distributive lattice with order–reversing involution" ' ". Let 0 be the least element and 1 be the greatest one in L. Suppose X is a nonempty (usual) set, an L-fuzzy set in X is a mapping A: $X\to L$, and L^X will denote the family of all L-fuzzy sets in X. It is clear that $L^X=\langle L^X,\leqslant,\wedge,\vee,'\rangle$ is a fuzzy lattice ,which has the least element $\underline{0}$ and the greatest one $\underline{1}$, where $\underline{0}(x)=0$, $\underline{1}(x)=1$,for any $x\in X$.

Definition1.1. A binary relation $\ll$ on L^X is called an L-fuzzy semi-topogenous order if it satisfies the following axiom: (1) $\underline{0}\ll\underline{0}$ and $\underline{1}\ll\underline{1}$; (2) $A\ll B$ implies $A\leqslant B$; (3) $A_1\leqslant A\ll B\leqslant B_1$ implies $A_1\ll B_1$. The complement of an L-fuzzy semi-topogenous order $\ll$ is the L-fuzzy semi-topogenous order $\ll^c$ defined by $A\ll^c B$ iff $B'\ll A'$. An L-fuzzy semi-topogenous order $\ll$ is called: (Ⅰ) symmetrical if $\ll=\ll^c$; (Ⅱ) topogenous if $A_1\ll B_1$ and $A_2\ll B_2$ implies $A_1\wedge A_2\ll B_1\wedge B_2$ and $A_1\vee A_2\ll B_1\vee B_2$; (Ⅲ) perfect if $Aj\ll Bj$, $j\in J$ implies $\vee Aj\ll\vee Bj$; (Ⅳ)co-perfect if $Aj\ll Bj$, $j\in J$ implies $\wedge Aj\ll\wedge Bj$; (Ⅴ)biperfect if perfect and co-perfect.

Suppose that $\ll_1,\ll_2$ are L-fuzzy semi-topogenous order on X, we call $\ll_1$ is finer than $\ll_2$ (i.e. $\ll_2$ is coarser than $\ll_1$) if for any $A,B\in L^X$, $A\ll_2 B$ implies $A\ll_1 B$, denoted by $\ll_2\leqslant\ll_1$. For a given L-fuzzy semi-topogenous order $\ll$, we define $\ll^p,\ll^1$ as follows: $A\ll^p B$ iff there exist Ai, $i\in I$, suth that $A=\vee Ai$, for any $i\in I$, $Ai\ll B$; $A\ll^1 B$ iff there exist Bj, $j\in J$ suth that $B=\wedge Bj$ and $A\ll Bj$, for each $j\in J$.

Definition1.2. An L-fuzzy syntopogenous structure on X is a nonempty family S of L-fuzzy semi-topogenous order on X having the following two properties:(LFS1) S is directed in the sense that given any two members of S there exist a member of S finer than both;(LFS2) For each $\ll$ in S there exist $\ll_1$ in S such that $A\ll B$ implies the existence of an L-fuzzy set C with $A\ll_1 C\ll_1 B$.

If S is an L-fuzzy syntopogenous structure on X, then the pair (X,S) is called an L-fuzzy syntopogenous space. An L-fuzzy syntopogenous structure S consisting of a single semi-topogenous order is called a topogenous structure and the pair (X,S) is called an L-fuzzy topogenous space. S is called perfect (resp. biperfect) if each member of S is perfect (resp. biperfect). An L-fuzzy syntopogenous structure S_1 is called finer than another one S_2, if for each $\ll$ in S_2 there exist a member of S_1 finer than $\ll$. In this case we also say that S_2 is coarser than S_1, denoted by $S_2 \leqslant S_1$. If S_1 is finer than S_2 and S_2 is finer than S_1, then S_1, S_2 are called equivalent, denoted by $S_1 \sim S_2$. To every L-fuzzy syntopogenous structure corresponds an L-fuzzy topology $\tau(S)$ given by the interior operator $\mu^0 = \sup\{\rho : \rho \ll \mu$ for some $\ll \in S\}$. If $\{\ll_\alpha : \alpha \in \Lambda\}$ is a family of L-fuzzy semi-topogenous order on X then $\ll = \vee_{\alpha\in\Lambda} \ll_\alpha$ is the L-fuzzy semi-topogenous order defined by $\mu \ll \rho$ iff $\mu \ll_\alpha \rho$ for some $\alpha \in \Lambda$. If S is a L-fuzzy syntopogenous structure, then it is easy to see that $\ll_s = \vee\{\ll : \ll \in S\}$ is an L-fuzzy topogenous order and $\{\ll_s\}$ is an L-fuzzy topogenous structure. Moreover, $\mu \in \tau(S)$ iff $\mu \ll^p_s \mu$. To every fuzzy topology τ on X corresponds a perfect L-fuzzy topogenous structure $S_\tau = \{\ll\}$, where $\mu \ll \rho$ iff there exists $\sigma \in \tau$ with $\mu \leqslant \sigma \leqslant \rho$. Moreover, $\tau = \tau(S)$. Conversely, to every perfect L-fuzzy topogenous structure $S = \{\ll\}$ corresponds the L-fuzzy topology $\tau = \tau(S)$, where $\mu \in \tau$ iff $\mu \ll \mu$. To two different L-fuzzy topologies correspond different perfect L-fuzzy topogenous structure[1][2].

2. *-increasing and *-decreasing spaces

A preorder on X is a binary relation"$\leqslant$"on X which is reflexive and transitive. preorder on X which is also anti-symmetric is called a partial order or simply an order. By a preordered (resp. an ordered) set we mean a set with a preorder (resp. a partial order) on it

Definition2.1. (Katsaras [4]) Let $(X, \leqslant)$ be a preorder set, $A \in L^X$ is called: (i) *-increasing if $x \leqslant y$ implies $A(x) \leqslant A(y)$; (ii) *-decreasing, if $x \leqslant y$ implies $A(y) \leqslant A(x)$; (iii) order-convex, if $y \leqslant x \leqslant z$ implies $A(y) \wedge A(z) \leqslant A(x)$.

Definition2.2. Let $(X, \leqslant)$ be a preorder set, define mappings p,a,c: $L^X \to L^X$ as follows: for any $A \in L^X$, $x \in X$, $p(A)(x) = \vee\{A(y): y \leqslant x$; $a(A)(x) = \wedge\{A(y) : x \leqslant y$; $c(A) = p(A) \wedge a(A)$.

Theorem2.1. Let (X,S) be an L-fuzzy syntopogenous space, define a binary relation $\leqslant_s$ on X as follows: for any $x, y \in X$, $x \leqslant_s y$ iff for $A \in L^X, \ll \in S, \lambda \in L, \lambda \neq 0$ and $x_\lambda \ll A$ implies $y_\lambda \leqslant A$, then "$\leqslant_s$" is a preorder on X, it is called the preorder generated by S on X.

proof.(1) (Reflexivity) We can get immediately from $x_\lambda \ll A$ implies $x_\lambda \leqslant A$.

(2) (Transitivity) Suppose $x \leqslant_s y$, $y \leqslant_s z$ and for $A \in L^X$, $\ll \in S$, $\lambda \in L, \lambda \neq 0$, $x_\lambda \ll A$. By (LFS_2) there exist $\ll_1 \in S$, $B \in L^X$, such that $x_\lambda \ll_1 B \ll_1 A$. Since $x \leqslant_s y$, thus $y_\lambda \leqslant B \ll_1 A$, so $y_\lambda \ll_1 A$. Also because $y \leqslant_s z$, hence $z_\lambda \leqslant A$, i.e. $x \leqslant_s z$.

Definition2.3. Let $(X, \leqslant)$ be a preorder set, S be an L-fuzzy syntopogenous structure on X, then $(X, S, \leqslant)$ is called *-increasing (*-decreasing) if for $x, y \in X$, $x \leqslant y$ implies $x \leqslant_s y$ ($y \leqslant_s x$).

Proposition2.2. (1) Let $S_1, S_2 \in S(X)$ and if $S_2 \leqslant S_1$, then for $x, y \in X$, $x \leqslant_{s1} y$ implies $x \leqslant_{s2} y$. And if $S_1 \sim S_2$, then $\leqslant_{s1} = \leqslant_{s2}$. (2) If $(X, S_1, \leqslant)$ is *-increasing (*-decreasing) and $S_2 \leqslant S_1$, then $(X, S_2, \leqslant)$ is *-increasing (*-decreasing).

Theorem2.3. If $\leqslant$ is a preorder on X, $S \in S(X)$, then (1) $(X, S, \leqslant_s)$ is *-increasing; (2) $(X, S_\leqslant, \leqslant)$ is

*-increasing; (3) $(X,S,\leqslant)$ is *-increasing iff $S\leqslant S_{\ll}$.

Proof. (1) Obvious.

(2) If $x\leqslant y$, $S_{\ll}=\{\langle\langle\}$ [6], for $A\in L^X$, $\lambda\in L$, $\lambda\neq 0$ and $x_\lambda \langle\langle A$ implies $x_\lambda(x)\leqslant A(y)$, i.e. $y_\lambda\leqslant A$. thus $x\leqslant_{S_{\ll}} y$, by Definition2.3 $(X,S_{\ll},\leqslant)$ is *-increasing.

(3)"$\Rightarrow$"If $S\leqslant S_{\ll}$, by (2) $(X,S_{\ll},\leqslant)$ is *-increasing, from Proposition2.2(2)then$(X,S,\leqslant)$is *-increasing.

"$\Leftarrow$"Suppos $\langle\langle_1\in S$, $S_{\ll}=\{\langle\langle\}$, $A\langle\langle_1 B$ and $x\leqslant y$, if $A(x)=0$, easily $A(x)\leqslant A(y)$. If $A(x)\neq 0$, because $x_{A(x)}\leqslant A\langle\langle_1 B$, then $x_{A(x)}\langle\langle_1 B$, as $(X,S,\leqslant)$ is *-increasing, so $y_{A(x)}\leqslant B$, i.e. $A(x)\leqslant B(y)$, thus $A\langle\langle B$, $S\leqslant S_{\ll}$.

Corollary2.4. Let $(X,\leqslant)$ be a preorder set, $H_i=\{E\in L^X$: E is increasing on $(X,\leqslant)\}$, define binary relation $\langle\langle_{H_i}$ as follows: $A\langle\langle_{H_i} B$ iff there exists $E\in H_i$ such that $A\leqslant E\leqslant B$. Then $S\in S(x)$, $(X,S,\leqslant)$ is *-increasing iff $S\leqslant\{\langle\langle_{H_i}\}$.

proof. Easily by Theorems 3.8[6] and 2.3.

Theorem2.5. The supremum of any number of *-increasing (*-decreasing) L-fuzzy syntopogenous structures on X is also *-increasing (*-decreasing)

The proof is omitted.

Corollary2.6. S^u (S^1) is the finest one of all *-increasing (*-decreasing) L-fuzzy syntopogenous structures which is coarser than S on S(X). Where $S^u=\vee\{S'\in S(X):(X, S',\leqslant)$ *-increasing, $S'\leqslant S\}$; $S^1=\vee\{S'\in S(X):(X, S',\leqslant)$ *-decreasing, $S'\leqslant S\}$. And (1) $S_1\leqslant S$ implies $S^u_1\leqslant S^u$, $S^1_1\leqslant S^1$; (2) $S_1\sim S$ implies $S_1{}^u\sim S^u$, $S^1{}_1\sim S^1$.

Proposition2.7. (1) If f: $(X,\leqslant)\rightarrow(Y,S,\leqslant')$ is an increasing mapping, $(Y,S,\leqslant')$ is *-increasing (*-decreasing), then $(X,f^{-1}(S),\leqslant)$ is *-increasing (*-decreasing); (2)If f is a decreasing mapping, $(Y,S,\leqslant')$ is *-increasing (*-decreasing), then $(X, f^{-1}(S),\leqslant)$ is *-decreasing (*-increasing).

Theorem2.8. If $\{(X_\lambda, S_\lambda, \leqslant_\lambda): \lambda\in\wedge\}$is a family of *-increasing (*-decreasing) L-fuzzy syntopogenous space, then the prodact $(\Pi_{\lambda\in\wedge}X_\lambda, \Pi_{\lambda\in\wedge}S_\lambda,\leqslant)$ is *-increasing (*-decreasing), where $\{x_\lambda\}\leqslant\{y_\lambda\}$ iff for any $\lambda\in\wedge$, $x_\lambda\leqslant_\lambda y_\lambda$.

Proof. By Corollary 2.6 (1) and Def. 7.1 .([4]).

3. a-order-convexity

Definition3.1. Let S be an L-fuzzy syntopogenous structure on $(X,\leqslant)$, $(X,S,\leqslant)$ will be said to be a-order-convex iff $S\sim(S^u\vee S^1)^a$, for $a\in\{i,p,b\}$ where i is identity.

Proposition3.1 If $(X,S,\leqslant)$ is a-order-convex, then $S\sim S^a$.

Proof. If $(X,S,\leqslant)$ is a-order-convex, then $S^a\sim(S^u\vee S^1)^{aa}=(S^u\vee S^1)^a\sim S$, so $S\sim S^a$.

Theorem3.2. $(X,S,\leqslant)$ is a-order-convex iff $S\sim(S_1\vee S_2)^a$, where $(X,S_1,\leqslant)$ ($(X,S_2,\leqslant)$) is *-decreasing (*-increasing).

Proof. The necessity is obvious. Conversely, if $S\sim(S_1\vee S_2)^a$, then$S_i\leqslant(S_1\vee S_2)\leqslant(S_1\vee S_2)^a\leqslant S$,(i=1,2) $S_1\leqslant S^u$, $S_2\leqslant S^1$, therefore $S\sim(S_1\vee S_2)^a\leqslant(S^u\vee S^1)^a\leqslant S^a$, but $S^a\sim(S_1\vee S_2)^{aa}=(S_1\vee S_2)^a\sim S$, so that $(S^u\vee S^1)^a\sim S$.

Proposition3.3. If $(X,S,\leqslant)$ is a-order-convex, $a'\in\{i,p,b\}$ is an elementary operation such that aa' is also an elementary operation, then $(X,S^{a'},\leqslant)$ is aa'-order-convex.

Proof. If $S\sim(S^u\vee S^1)^a$, then $S^{a'}\sim(S^u\vee S^1)^{aa'}$. From Theorem 3.2 have $(X, S^{a'},\leqslant)$ is aa'-order-convex.

Theorem3.4. If $(X,S,\leqslant)$ is a-order-convex, then $(X,S^{ta},\leqslant)$ is also a-order-convex.

Proof. If $(X,S,\leqslant)$ is a-order-convex, then $S^{ta}\sim(S^u\vee S^1)^{ata}=(S^u\vee S^1)^{ta}\sim(S^u\vee S^1)^a\sim(S^{tu}\vee S^{t1})^a$, by Theorem 3.2 $(X,S^{ta},\leqslant)$ is a-order-convex..

Theorem3.5. Let $\{S_i: i\in I\neq\Phi\}$ be a family of a-order-convex L-fuzzy syntopogenous structure on the preorder set $(X,\leqslant)$, then $(\vee_{i\in I}S_i)^a$ is also a-order-convex on $(X,\leqslant)$.

Proof. Put $S=(\vee_{i\in I}S_i)^a$, $S_1=\vee_{i\in I}S_i^u$ and $S_2=\vee_{i\in I}S_i^1$, then S_1 is *-increasing, S_2 is *-decreasing on $(X,\leqslant)$. As $S_i\sim(S_i^u\vee S_i^1)^a$, then $(\vee_{i\in I}S_i)^a\sim(\vee_{i\in I}(S_i^u\vee S_i^1)^a)^a\sim((\vee_{i\in I}S_i^u)\vee(\vee_{i\in I}S_i^1))^a\sim(S_1\vee S_2)^a$, $S\sim(S_1\vee S_2)^a$, by Theorem 3.2 $(\vee_{i\in I}S_i)^a$ is also a-order-convex on $(X,\leqslant)$.

Theorem3.6. Let $(X,\leqslant)$, $(X',\leqslant')$ be preordered set, f is a preorder preserving mapping from X to X'. If $(X', S',\leqslant')$ is a-order-convex, then $(X, f^1(S'),\leqslant)$ is also a-order-convex.

Proof. If $(X', S',\leqslant')$ is a-order-convex, then $f^1(S')\sim f^1((S'^u\vee S'^1)^a)=f^1(S'^u\vee S'^1)^a=(f^1(S'^u)\vee f^1(S'^1))^a$, by Proposition2.7, $f^1(S'^u)$ is *-increasing, $f^1(S'^1)$ is *-decreasing on $(X,\leqslant)$. Also by Theorem 3.2 then $(X,f^1(S'),\leqslant)$ is also a-order-convex.

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