The fuzzy-valued Choquet integral of functions with respect to fuzzy-valued fuzzy measures

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It is well-known that the theory of fuzzy-valued integrals[6], fuzzy-valued fuzzy integrals[2] have been built up, it is necessary to study the Choquet integral[1] valued in fuzzy numbers. In this paper, a theory of Choquet integral of functions with respect to fuzzy-valued fuzzy measures will be shown.

Throughout the paper, the following symbols and notations will be used. X is a classical set, A is a σ-algebra formed by the subsets of X, (X, A) is the measurable space, $F(X)$ is the set of all nonnegative measurable functions from X to $R^+=[0, \infty)$, $\mu:\mathsf{A}\to R^+$ is a fuzzy measure under Sugeno' sense, (c) $\int_A f d\mu$ is the Choquet integral, $M(X)$ denotes the set of all fuzzy measures on (X, A).

We adopt the concepts about interval number, fuzzy number, interval-number fuzzy measure, fuzzy number fuzzy measure, in [2] . For the sake of brief, they are omitted

Definition 1 Let $f \in F(X)$, $A \in \mathsf{A}$, $\bar{\mu} \in \bar{M}(X)$, then the Choquet

integral of f on A with respect to $\bar{\mu}$ is defined as

$$(c)\int_A f d\bar{\mu} = [(c)\int_A f d\mu^-, (c)\int_A f d\mu^+]$$

Definition 2 Let $f \in F(X)$, $A \in \mathbf{A}$, $\tilde{\mu} \in \tilde{M}(X)$. Then the Choquet integral of f on A with respect to $\tilde{\mu}$ is defined as

$$\left((c)\int_A f d\tilde{\mu}\right)(r) = \sup\left\{\lambda \in (0,1] : r \in (c)\int_A f d\bar{\mu}_\lambda\right\}.$$

If $(c)\int_A f d\tilde{\mu} < \infty$, then f is said to be integrable on A with respect to $\tilde{\mu}$

Theorem 1 Let $f \in F(X)$, $A \in \mathbf{A}$, $\tilde{\mu} \in \tilde{M}(X)$. If f is integrable on A with respect to $\tilde{\mu}$, then $(c)\int_A f d\tilde{\mu} \in \tilde{R}^+$, and

$$\left((c)\int_A f d\tilde{\mu}\right)_\lambda = \int_A f d\bar{\mu}_\lambda, \quad \lambda \in (0,1]$$

Theorem 2 The Choquet integral of functions with respect to fuzzy number fuzzy measures has the following properties:

(i) $f \le g$ implies $(c)\int_A f d\tilde{\mu} \le (c)\int_A g d\tilde{\mu}$;

(ii) $A \subset B$ implies $(c)\int_A f d\tilde{\mu} \le (c)\int_B f d\tilde{\mu}$;

(iii) $(c)\int_A af d\tilde{\mu} = a(c)\int_A f d\tilde{\mu}$;

(iv) If $\tilde{\mu}$ is a additive fuzzy number measure[11], then the fuzzy-valued Choquet integral coincides with the fuzzy-valued Lebesgue integral[11].

$$(c)\int_A f d\tilde{\mu} = (L)\int_A f d\tilde{\mu}$$

(v) $(c)\int_A I_A d\tilde{\mu} = \tilde{\mu}(A)$, $A \in \mathbf{A}$, I_A denotes the characteristic function of A.

Corollary 1

(i) $(c)\int_A (f \vee g) d\tilde{\mu} \geq (c)\int_A f d\tilde{\mu} \vee (c)\int_A g d\tilde{\mu}$;

(ii) $(c)\int_A (f \wedge g) d\tilde{\mu} \geq (c)\int_A f d\tilde{\mu} \wedge (c)\int_A g d\tilde{\mu}$;

(iii) $(c)\int_{A \cup B} f d\tilde{\mu} \geq (c)\int_A f d\tilde{\mu} \vee (c)\int_B f d\tilde{\mu}$;

(iv) $(c)\int_{A \cap B} f d\tilde{\mu} \leq (c)\int_A f d\tilde{\mu} \wedge (c)\int_B f d\tilde{\mu}$;

Definition 3 A set $N \in$ A is called a null-set with respect to $\tilde{\mu}$ iff $\tilde{\mu}(A \cup N) = \tilde{\mu}(A)$, for all A $\in$ A.

Theorem 3 If W^c is a null set, then for every measurable function f,

$$(c)\int_A f d\tilde{\mu} = (c)\int_A f_w d\tilde{\mu}_w$$

Where f_w is the restriction of f on W, and $\tilde{\mu}_w$ is similar.

Theorem 4 Give a measurable set N the following conditions are equivalent,

(i) N is a null set;

(ii) $(c)\int_X f d\mu = (c)\int_X g d\mu$, for all $f, g \in F(X)$, such that $f(x) = g(x)$; $\forall x \in N^c$

Theorem 5 Let $\{f_n (n \geq 1), f\} \subset F(X), \{\tilde{\mu}_n (n \geq 1), \tilde{\mu}\} \subset \tilde{M}(X)$. If $f_n \uparrow f, \tilde{\mu}_n \uparrow \tilde{\mu}$, then

$$(c)\int_A f_n d\tilde{\mu}_n \uparrow (c)\int_A f d\tilde{\mu}$$

Theorem 6 Let $\{f_n (n \geq 1), f\} \subset F(X), \{\tilde{\mu}_n (n \geq 1), \tilde{\mu}\} \subset \tilde{M}(X)$. If there exists a $n_0 \in N$, such that $(c)\int_A f_{n_0} d\tilde{\mu}_{n_0} < \infty$, and $f_n \downarrow f, \tilde{\mu}_n \downarrow \tilde{\mu}$,

then

$$(c)\int_A f_n d\tilde{\mu}_n \downarrow (c)\int_A f d\tilde{\mu}$$

Corollary 2 Let $\{f_n(n \geq 1), f\} \subset F(X), \tilde{\mu} \in \tilde{M}(X)$.

(i) If $f_n \uparrow f$, then

$$(c)\int_A f_n d\tilde{\mu} \uparrow (c)\int_A f d\tilde{\mu}$$

(ii) If there exists a $n_0 \in N$, such that $(c)\int_A f_{n_0} d\tilde{\mu}_{n_0} < \infty$, and $f_n \downarrow f$, then

$$(c)\int_A f_n d\tilde{\mu} \downarrow (c)\int_A f d\tilde{\mu}$$

Concluding remark. Up to here, the theory of the Chqouet integral of functions with respect to fuzzy number fuzzy measures is built up. It is a kind of fuzzy-valued Choquet integral, which made the number valued Choquet integral as a special case. In a subsequent one, we will show the theory of Choquet integral of fuzzy-valued functions with respect to fuzzy number fuzzy number fuzzy measures.

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