Aggregations preserving classes of fuzzy relations *

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Abstract

Using aggregation functions of n variables we consider aggregations of fuzzy relations. After recalling fundamental properties of fuzzy relations we examine aggregation functions which preserve some of these properties.

Keywords: aggregation function, quasi arithmetic mean, generalized weighted average, fuzzy relation, relation classes, fuzzy equivalence relation, fuzzy order relation.

1 Introduction

Since L.A. Zadeh [9], [10] has introduced the definition of fuzzy relations, the theory of them was developed by several authors. In particular, internal unary and binary operations in classes of fuzzy relations were described by Drewniak [4]. Recently, in connection with multicriteria decision making, n-ary aggregation operations are examined (cf. Peneva, Popchev [8]). The question arises if these aggregations preserve properties of aggregated fuzzy relations.

Definition 1 (Calvo, Mayor [2]). Let $n \geq 2$. $A : \mathbb{R}^n \to \mathbb{R}$ is an aggregation function if it is increasing and idempotent, i.e.

$$\forall_{s,t \in \mathbb{R}^n} \left(\forall_{1 \leq k \leq n} s_k \leq t_k \right) \Rightarrow A(s_1, \cdots, s_n) \leq A(t_1, \cdots, t_n), \tag{1}$$

$$\forall_{t \in \mathbb{R}} A(t, \cdots, t) = t. \tag{2}$$

*Presented on 'The sixth Internat. Conf. Soft Computing & Distributed Processing', June 24-25, 2002, Rzeszów, Poland.

Example 1. Let $\varphi : [0,1] \to [0,1]$ be an increasing bijection and let $t, w \in [0,1]^n$, $\sum_{k=1}^{n} w_k = 1$. We remind there two important examples of aggregation functions: the quasi arithmetic mean (cf. Aczél [1])

$$A(t_1, \cdots, t_n) = \varphi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}\varphi(t_k)\right), \tag{3}$$

and the generalized weighted average (cf. Calvo, Mayor [2])

$$A(t_1, \cdots, t_n) = \varphi^{-1}\left(\sum_{k=1}^{n} w_k\varphi(t_k)\right). \tag{4}$$

Definition 2 (Zadeh [9]). Let $X \neq \emptyset$. A fuzzy relation in X is an arbitrary function $R : X \times X \to [0,1]$. The family of all fuzzy relations in X is denoted by FR(X).

Definition 3 (Peneva, Popchev [7]). Let A be an aggregation function and $R_1, \cdots, R_n \in FR(X)$. By an aggregation fuzzy relation we call $R \in FR(X)$,

$$R(x,y) = A(R_1(x,y), \cdots, R_n(x,y)),\ x, y \in X. \tag{5}$$

We shall examine properties of the relation (5) under suitable assumptions on aggregation A and fuzzy relations $R_1, \cdots, R_n$.

2 Classes of fuzzy relations

Now we remind fuzzy versions of known relation properties.

Definition 4 (cf. Drewniak [3]). A fuzzy relation $R \in FR(X)$ is called

- reflexive if $\forall_{x \in X}\ R(x,x) = 1$,
- weakly reflexive if $\forall_{x \in X}\ R(x,x) > 0$,
- irreflexive if $\forall_{x \in X}\ R(x,x) = 0$,
- weakly irreflexive if $\forall_{x \in X}\ R(x,x) < 1$,
- symmetric if $\forall_{x,y \in X}\ R(y,x) = R(x,y)$,
- weakly symmetric if $\forall_{x,y \in X}\ R(x,y) = 1 \Rightarrow R(y,x) = 1$,

- semi-symmetric if $\underset{x,y\in X}{\forall} R(x,y) > 0 \Rightarrow R(y,x) > 0$,
- asymmetric if $\underset{x,y\in X}{\forall} R(x,y) > 0 \Rightarrow R(y,x) = 0$,
- weakly asymmetric if $\underset{x,y\in X}{\forall} R(x,y) = 1 \Rightarrow R(y,x) < 1$,
- antisymmetric if $\underset{x,y\in X, x\neq y}{\forall} R(x,y) > 0 \Rightarrow R(y,x) = 0$,
- weakly antisymmetric if $\underset{x,y\in X, x\neq y}{\forall} R(x,y) = 1 \Rightarrow R(y,x) < 1$,
- complete if $\underset{x,y\in X}{\forall} R(x,y) < 1 \Rightarrow R(y,x) = 1$,
- weakly complete if $\underset{x,y\in X}{\forall} R(x,y) = 0 \Rightarrow R(y,x) > 0$,
- transitive if $\underset{x,y,z\in X}{\forall} R(x,z) \geq \min(R(x,y), R(y,z))$,
- weakly transitive if $\underset{x,y,z\in X}{\forall} \min(R(x,y), R(y,z)) > 0 \Rightarrow R(z,x) > 0$.

It is evident that the characteristic function of a crisp binary relation with suitable property fulfils the adequate condition. Using an arbitrary binary operation $* : [0,1]^2 \to [0,1]$ we can consider diverse transitivity properties.

Definition 5 (cf. Goguen [6], Fodor [5]). A fuzzy relation R is called $*$-transitive or weakly $*$-transitive if respectively fulfils

$$\underset{x,y,z\in X}{\forall} R(x,z) \geq R(x,y) * R(y,z), \tag{6}$$

$$\underset{x,y,z\in X}{\forall} R(x,y) * R(y,z) > 0 \Rightarrow R(z,x) > 0. \tag{7}$$

The above properties can be combined together in order to obtain new classes of fuzzy relations. As an example we put.

Definition 6 (cf. Zadeh [10], Drewniak [3]). $R \in FR(X)$ is called fuzzy equivalence ($*$-equivalence) relation if it is reflexive, symmetric and transitive ($*$-transitive). For weak properties it is called fuzzy weak equivalence. R is called fuzzy order ($*$-order) relation if it is reflexive, antisymmetric and transitive ($*$-transitive). For weak properties it is called fuzzy weak order.

Example 2. Let $card\, X = 3$ and fuzzy relations $R, S \in FR(X)$ have matrices

$$R = \begin{bmatrix} 1 & 0{,}6 & 0{,}7 \\ 0{,}6 & 1 & 0{,}6 \\ 0{,}7 & 0{,}6 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0,5 & 0,8 \\ 0 & 1 & 0,5 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can check that $R^2 = R$, $S^2 = S$ and R is a fuzzy equivalence relation, and S is a fuzzy order relation.

3 Results

Now we are able to present properties of the aggregation relation (5).

Theorem 1. *Every aggregation function preserves reflexivity, weak reflexivity, irreflexivity, weak irreflexivity and symmetry of fuzzy relations.*

Theorem 2. *Every generalized weighted average preserves weak symmetry, semi-symmetry, asymmetry, weak asymmetry, antisymmetry, weak antisymmetry and weak completeness of fuzzy relations.*

Example 3. For certain properties in the above theorem the aggregation function cannot be arbitrary. Let $n = 2$ and $card\, X = 2$,

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, max(R, S) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Fuzzy relations R, S are asymmetric, but $max(R, S)$ is a not asymmetric fuzzy relation. Similarly we have

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad min(R, S) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where relations R, S are complete but $min(R, S)$ is not complete. In the case of transitivity property even the arithmetic mean does not preserve it. We have

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \frac{R + S}{2} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix},$$

where relations R and S are transitive, but fuzzy relation $\frac{R+S}{2}$ is not transitive.

Theorem 3 (cf. Peneva, Popchev [7]). *Let* $* : [0, 1]^2 \to [0, 1]$ *be the Lukasiewicz multivalued conjunction:*

$$x * y = max(0, x + y - 1), \; x, y \in [0, 1]. \tag{8}$$

The weighted average preserves $*$-transitivity and weak $*$-transitivity.

Similar results are not possible for the generalized weighted average (4) or for the quasi-arithmetic mean (3) with arbitrary bijection $\varphi \neq Id$.

Example 4. Let $n = 2$, $card\, X = 3$, $\varphi(x) = x^2$, $x \in [0,1]$. Using operation (8) relations S, T with matrices

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \tag{9}$$

are $*$-transitive (and weakly $*$-transitive). However, fuzzy relation $R = [r_{ik}]$,

$$r_{ik} = \sqrt{\frac{s_{ik}^2 + t_{ik}^2}{2}},\ i,k = 1,2,3, \quad R = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

is not $*$-transitive and even weakly $*$-transitive. The result of Theorem 3 is not valid for arbitrary binary operation other than (8). Let us observe that relations (9) are product-transitive (and weakly product-transitive) with ordinary product in $[0,1]$. However, fuzzy relation $R = \frac{S+T}{2}$ is not product-transitive and even weakly product-transitive, because

$$R = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad 0 = r_{12} < r_{13} r_{32} = \frac{1}{4}.$$

From the above results we get (cf. Peneva, Popchev [7])

Theorem 4. *The weighted average preserves $*$-equivalence, weak $*$-equivalence and weak $*$-order of fuzzy relations with the operation* (8).

4 Concluding remarks

Our results can be presented in more general form if we change the main question. Instead of looking for aggregations preserving given relation property, we can ask about assumptions sufficient for this property of aggregation function. E.g., if we consider quasi-arithmetic mean and one of considered fuzzy relations is weakly reflexive, then their aggregation is also weakly reflexive. However, this is a question for another paper.

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