

A characterization of idempotent nullnorms *

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Abstract

Nullnorm $V : [0, 1]^2 \rightarrow [0, 1]$ is a generalization of triangular norm and triangular conorm allow lying a zero element anywhere in the unit interval. Nullnorms are built up from triangular norms and triangular conorms. If nullnorm is idempotent, then triangular norm is equal to minimum, and triangular conorm is equal to maximum. The idea of this paper is to find assumptions for operation in order to get idempotent nullnorm.

Keywords: Nullnorm, triangular norm, triangular conorm, binary operations, increasing operation, idempotent operation, zero element.

1 Introduction

Nullnorm is a generalization of triangular norm and triangular conorm allow lying a zero element anywhere in the unit interval, and have to satisfy an additional condition. Such nullnorms are interesting not only from a theoretical point of view (because of their structure as combinations of a triangular norm and a triangular conorm), but also for their applications, since they have proved to be useful in several fields as expert systems, neural networks, fuzzy quantifiers (cf. [4]). Moreover nullnorms that have to be used as aggregators *or* in fuzzy logic maintain the most logical properties as possible.

We discuss the structure of binary operations $V : [0, 1]^2 \rightarrow [0, 1]$.

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Definition 1 ([1]). Operation T is called triangular norm (t -norm) if it is commutative, associative, increasing, and has a neutral element $e = 1$, i.e.

$$T(1, x) = x \text{ for } x \in [0, 1]. \quad (1)$$

Operation S is called triangular conorm (t -conorm) if it is commutative, associative, increasing, and has a neutral element $e = 0$, i.e.

$$S(0, x) = x \text{ for } x \in [0, 1]. \quad (2)$$

By monotonicity from (1) and (2) we get

Corollary 1. Triangular norm has a zero element $z = 0$, triangular conorm has a zero element $z = 1$.

Definition 2 ([1]). Operation V is called nullnorm if it is commutative, associative, increasing, has a zero element $z \in [0, 1]$, and satisfies

$$V(0, x) = x \text{ for all } x \leq z, \quad (3)$$

$$V(1, x) = x \text{ for all } x \geq z. \quad (4)$$

By definition, the case $z = 0$ leads back to t -norms, while the case $z = 1$ leads back to t -conorms (cf. [3]), because in the first case we have by (4) that one is neutral element in the unit interval, and in the second case by (3) that zero is neutral element in the unit interval.

If in definition of nullnorm we omit assumptions (3) and (4), it cannot be shown that a commutative, associative, increasing binary operation V with zero element $z = 0$ or $z = 1$ behaves as a t -norm and t -conorm.

Example 1. Let $z \in [0, 1]$. We define the operation as follows $V(x, y) = z$. The operation is associative, increasing, commutative, with zero element z , but if $z = 0$ or $z = 1$ we not obtain a t -norm or t -conorm.

2 Structure of nullnorms

The next theorem show that nullnorm is built up from a t -norm, a t -conorm and the value of the zero element.

Theorem 1 ([1]). Let $z \in (0, 1)$. A binary operation V is a nullnorm with zero element z if and only if there exists triangular norm T and triangular conorm S such that

$$V = \begin{cases} S^* \text{ in } [0, z]^2 \\ T^* \text{ in } [z, 1]^2 \\ z \text{ otherwise} \end{cases}, \quad (5)$$

where

$$\begin{cases} S^*(x, y) = \varphi^{-1}(S(\varphi(x), \varphi(y))), \varphi(x) = x/z, & x, y \in [0, z] \\ T^*(x, y) = \psi^{-1}(T(\psi(x), \psi(y))), \psi(x) = (x - z)/(1 - z), & x, y \in [z, 1] \end{cases}. \quad (6)$$

Lemma 1 ([2]). If increasing operation have zero element $z \in [0, 1]$, then

$$V(x, y) = z \text{ for } (x, y) \in [0, z] \times [z, 1] \cup [z, 1] \times [0, z].$$

In Definition 2 we can omit the existence of zero element z because it follows from (3) and (4):

Lemma 2. Let V be an increasing binary operation. If exist $z \in [0, 1]$ such that

$$V(0, x) = V(x, 0) = x \text{ for all } x \leq z, \quad (7)$$

$$V(1, x) = V(x, 1) = x \text{ for all } x \geq z, \quad (8)$$

then z is a zero element of V , and

$$V(x, y) \geq \max(x, y) \text{ for } x, y \in [0, z], \quad (9)$$

$$V(x, y) \leq \min(x, y) \text{ for } x, y \in [z, 1]. \quad (10)$$

Corollary 2. Under assumptions of Lemma 2 $V|_{[0, z]}$ is increasing, binary operation with neutral element 0 and zero element z . $V|_{[z, 1]}$ is increasing, binary operation with neutral element 1 and zero element z .

Moreover V is associative (commutative, idempotent) iff $V|_{[0, z]}$ and $V|_{[z, 1]}$ are associative (commutative, idempotent).

3 Idempotent nullnorms

First we remind the definition of idempotent operations, and structure of idempotent nullnorms.

Definition 3. *Operation V is called idempotent if*

$$V(x, x) = x \text{ for all } x. \quad (11)$$

Theorem 2 (cf. [2]). *Let $z \in [0, 1]$. A continuous, idempotent, associative and increasing binary operation V satisfies*

$$V(0, 1) = V(1, 0) = z \quad (12)$$

iff it is given by

$$V = \begin{cases} \max & \text{in } [0, z]^2 \\ \min & \text{in } [z, 1]^2 \\ z & \text{otherwise} \end{cases}. \quad (13)$$

In the above theorem conditions (7) and (8) follow from assumption that V is continuous, idempotent, associative, increasing and satisfies (12). We ask what are the weakest assumption for V to have the form (13).

Lemma 3. *Let V be an increasing binary operation. If exists $z \in [0, 1]$ such that*

$$V(x, x) \leq x \text{ for all } x \leq z, \quad (14)$$

$$V(x, x) \geq x \text{ for all } x \geq z, \quad (15)$$

then

$$V(x, y) \leq \max(x, y) \text{ for } x, y \in [0, z], \quad (16)$$

$$V(x, y) \geq \min(x, y) \text{ for } x, y \in [z, 1]. \quad (17)$$

From Lemmas 2 and 3 follows that:

Theorem 3. *Let $z \in [0, 1]$. V is an increasing binary operation fulfilling (7), (8), (14), and (15), iff V is given by (13).*

Corollary 3. *If in the above theorem we replace (14) and (15) with idempotency condition (11), then we obtain the same form of V .*

Now we consider assumptions weaker then (7) and (8).

Lemma 4. *Let V be an increasing binary operation. If exist $z \in [0, 1]$ such that*

$$V(0, x) \geq x, \quad V(x, 0) \geq x \quad \text{for all } x \leq z, \quad (18)$$

$$V(1, x) \leq x, \quad V(x, 1) \leq x \quad \text{for all } x \geq z, \quad (19)$$

then z is a zero element of V , and we get (9) and (10).

Corollary 4. *Let $z \in [0, 1]$. If an increasing binary operation V fulfils conditions (18) and (19) for $x = z$, then z is a zero element of V .*

From Lemmas 3 and 4 follows

Theorem 4. *Let $z \in [0, 1]$. V is an increasing binary operation such that (14), (15), (18), (19) holds, iff V is given by (13).*

Corollary 5. *If in the above theorem we replace (14) and (15) with idempotency condition (11), then we obtain the same form of V .*

All assumptions of the above theorem are necessary. If we omit one of them, we can find suitable operations different from (13).

Example 2. *Let $z \in (0, 1)$, $x, y \in [0, 1]$ and*

$$\begin{aligned} V_1 &= \begin{cases} \min & \text{in } [z, 1]^2 \\ \max & \text{otherwise} \end{cases}, \quad V_2 = \begin{cases} \min & \text{in } [z, 1]^2 \\ z & \text{otherwise} \end{cases}, \\ V_3 &= \begin{cases} \max & \text{in } [0, z]^2 \\ z & \text{otherwise} \end{cases}, \quad V_4 = \begin{cases} \min & \text{in } [0, z]^2 \cup [z, 1]^2 \\ z & \text{otherwise} \end{cases}, \\ V_5(x, y) &= \begin{cases} \max(x, y) & \text{if } 0 \leq y \leq x < z \\ \frac{x+y}{2} & \text{if } 0 \leq x < y < z \\ z & \text{if } (x, y) \in [0, z] \times [z, 1] \cup [z, 1] \times [0, z] \\ \min(x, y) & \text{if } (x, y) \in (z, 1)^2 \end{cases}, \\ V_6(x, y) &= \begin{cases} \frac{x+y}{2} & \text{if } 0 \leq y < x < z \\ \max(x, y) & \text{if } 0 \leq x \leq y < z \\ z & \text{if } (x, y) \in [0, z] \times [z, 1] \cup [z, 1] \times [0, z] \\ \min(x, y) & \text{if } (x, y) \in (z, 1)^2 \end{cases}, \\ V_7 &= \begin{cases} \max & \text{in } [0, z]^2 \cup [z, 1]^2 \\ z & \text{otherwise} \end{cases}, \end{aligned}$$

$$V_8(x, y) = \begin{cases} \frac{x+y}{2} & \text{if } z < y < x \leq 1 \\ \min(x, y) & \text{if } z < x \leq y \leq 1 \\ z & \text{if } (x, y) \in [0, z] \times [z, 1] \cup [z, 1] \times [0, z] \\ \max(x, y) & \text{if } (x, y) \in [0, z]^2 \end{cases} ,$$

$$V_9(x, y) = \begin{cases} \min(x, y) & \text{if } z < y \leq x \leq 1 \\ \frac{x+y}{2} & \text{if } z < x < y \leq 1 \\ z & \text{if } (x, y) \in [0, z] \times [z, 1] \cup [z, 1] \times [0, z] \\ \max(x, y) & \text{if } (x, y) \in [0, z]^2 \end{cases} .$$

Operations $V_1 - V_9$ omit exactly one of consecutive assumptions from Theorem 4: monotonicity, (14), (15), (18), (18a), (18b), (19), (19a), (19b), where a) and b) concern left and right hand conditions.

Moreover operation $V_1 - V_4$ and V_7 are associative and commutative.

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