L-Fuzzy para - metacompact spaces and Perfect maps

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Abstract

Some good extensions of both para and metacompact spaces and their behavior under various kinds of maps are discussed in the L-fuzzy setup.

1.Introduction

The concept of paracompactness in fuzzy topology was introduced by Luo [3] in [0,1] set up. Some detailed sketches of different types of paracompactness in the L-fuzzy set up can be seen in [4]. Regarding metacompactness, author has some work in [0,1] case in [8]. Here we try to extent the investigation in to *L-Fts* using α–Q–covers and quasi coincidence relation. In [7] author has some results regarding the behavior of metacompact spaces under perfect maps in the [0,1] fuzzy context. Some work related to paracompactness and perfect maps can be found in [5]. In this paper, we also try to bring out the behaviour of para-meta type spaces under perfect maps in the L-fts. For defining the perfect maps, we make use of the concept of N- Compactness. Since compactness in fuzzy topology is defined in various forms, we can define different types of perfect maps correspondingly. The reason for choosing N-compactness is that it possesses better properties among all these types and it is defined in terms of α–Q–covers and quasi coincidence relation which are used in defining para-metacompact spaces in L-fts.

The lattice L we are considering is a complete, completely distributive one equipped with an order reversing involution. For basic definitions and notations, we

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follow Ying-Ming and Mao-Kang [4]. We take q to denote the quasi coincidence relation. Also χ denote the characteristic function and $Pt\,(L^X)$ is the collection of all L-fuzzy points in the *L-fts* (L^X,δ). A molecule in a lattice L is a join irreducible element in L and the set of all molecules of L is denoted by $M(L)$.

2. Preliminaries and Basic Definitions

2.1 Definition [4] Let L^X, L^Y be two L-fuzzy spaces, f: X→Y be an ordinary mapping. Based on this we define the L-fuzzy mapping $f^{\rightarrow}: L^X \to L^Y$ and its L-fuzzy reverse mapping $f^{\leftarrow}: L^Y \to L^X$ by

$f^{\rightarrow}: L^X \to L^Y$, $f^{\rightarrow}(A)(y) = \vee\{A(x) : x\in X, f(x) = y\}$ $\forall A\in L^X$, $\forall y\in Y$.

$f^{\leftarrow}: L^Y \to L^X$, $f^{\leftarrow}(B)(x) = B(f(x))$, $\forall B\in L^Y$, $\forall x\in X$.

2.2 Definition [4] Let (L^X,δ), (L^Y,μ) be *L-fts's*, $f^{\rightarrow}: L^X \to L^Y$ an L-fuzzy mapping. We say $f^{\rightarrow}$ is an L-fuzzy continuous mapping from (L^X,δ) to (L^Y,μ) if its L-fuzzy reverse mapping $f^{\leftarrow}: L^Y \to L^X$ maps every open subset in (L^Y,μ) as an open one in (L^X,δ). ie, $\forall\, V\in \mu$, $f^{\leftarrow}(V) \in \delta$.

2.3 Definition [4] Let (L^X,δ), (L^Y,μ) be *L-fts's*, $f^{\rightarrow}: L^X \to L^Y$ an L-fuzzy mapping. We say $f^{\rightarrow}$ is open if it maps every open subset in (L^X,δ) as an open one in (L^Y,μ). ie, $\forall\, U\in \delta$, $f^{\rightarrow}(U) \in\mu$.

2.4 Definition [4] Let (L^X,δ), (L^Y,μ) be *L-fts's*, $f^{\rightarrow}: L^X \to L^Y$ an L-fuzzy mapping. We say $f^{\rightarrow}$ is closed if it maps every closed subset in (L^X,δ) as a closed one in (L^Y,μ). ie, $\forall\, F\in \delta'$, $f^{\rightarrow}(F) \in\mu'$.

2.5 Definition [4] Let (L^X,δ) be an *L-Fts*. Then by $[\delta]$ we denote the family of support sets of all crisp subsets in δ. (X, $[\delta]$) is a topology and it is the background space. (L^X,δ) is weakly induced if each $U\in\delta$ is a lower semi continuous function from the background space $(X, [\delta])$ to L.

2.6 Definition [4] For a property P of ordinary topological space, a property P^* of *L-Fts* is called a good L-extension of P, if for every ordinary topological space (X,T), (X,T) has the property P if and only if $(X,\omega_L(T))$ has property P^*. In particular when $L = [0,1]$ we say P^* is a good extension of P. Where $\omega_L(T)$ is the family of all lower semi continuous functions from (X,T) to L .

2.7 Definition [4] Let (L^X,δ) be an L-Fts . A fuzzy point x_α is quasi coincident with $A\in L^X$ (and write $x_\alpha \prec A$) if $x_\alpha \nleq A'$. Also A quasi coincides with B at x (AqB at x) if $A(x) \nleq B'(x)$. We say A quasi coincident with B and write AqB if AqB at x for some $x\in X$. Further $A\neg qB$ means A not quasi coincides with B. We say $U\in\delta$ is a quasi coincident *nbd* of x_α (*Q-nbd*) if $x_\alpha \prec U$. The family of all *Q-nbd*s of x_α is denoted by $Q_\delta(x_\alpha)$ or $Q(x_\alpha)$.

2.8 Definition [4] Let (L^X,δ) be an L-Fts , $A\in L^X$. $\Phi\subset L^X$ is called a Q-cover of A if for every $x\in Supp(A)$, there exists $U\in\Phi$ such that $x_{A(x)}\prec U$. Φ is a Q-cover of (L^X,δ) if Φ is a Q-cover of $\underline{1}$. If $\alpha\in M(L)$, then $C\in\delta$ is an α-Q-nbd of A , if $C\in Q(x_\alpha)$ for every $x_\alpha\leq A$. Φ is called an α-Q cover of A if for every $x_\alpha\leq A$, there exists $U\in\Phi$ such that $x_\alpha\prec U$. Φ is called an open α-Q-cover of A if $\Phi\subset\delta$ and Φ is an α-Q-cover of A. $\Phi_0\subset L^X$ is called a sub α-Q cover of A, if $\Phi_0\subset\Phi$ and Φ_0 is also an α-Q cover of A. Φ is called an $\bar{\alpha}$-Q-cover of A , if there exists $\gamma\in\beta^*(\alpha)$ such that Φ is a γ-Q cover of A.

2.9 Definition [4] Let (L^X,δ) be an L-Fts , $A\in L^X$. A is called N-compact if for every $\alpha\in M(L)$, every open α-Q cover of A has a finite subfamily which is an $\bar{\alpha}$-Q-cover of A. (L^X,δ) is called N-compact if $\underline{1}$ is N-compact.

2.10 Definition Let (L^X,δ) , (L^Y,μ) be L-fts's , $f^{\rightarrow}: L^X\rightarrow L^Y$ an L-fuzzy mapping. We say $f^{\rightarrow}$ is perfect if it is continuous ,closed and $f^{\leftarrow}(y)$ N-compact for every $y\in Y$.

2.11 Definition [4] Let (L^X,δ) be an L-Fts. $A=\{A_t: t\in T\}\subset L^X$, $x_\lambda\in M(L^X)$. A is called locally finite at x_λ if there exists $U\in Q(x_\lambda)$ and a finite subset T_0 of T such that $t\in T\setminus T_0 \Rightarrow A_t\neg qU$. And A is *-locally finite at x_λ if $t\in T\setminus T_0 \Rightarrow \chi_{At_{(0)}}\neg qU$. where $At_{(0)}=\{x\in X:$ $A_t(x)\quad 0\}$. A is called locally finite (resp.*-locally finite) for short , if A is locally finite (resp. *-locally finite) at every molecule x_λ of L^X.

2.12 Definition [4] Let (L^X,δ) be an L-Fts. $A\in L^X$, $\alpha\in M(L)$. A is called α-paracompact (resp.α^*-paracompact) if for every open α-Q-cover Φ of A, there exists an open refinement Ψ of Φ such that Ψ is locally finite (resp. *-locally finite)in A and Ψ is also an α-Q-cover of A. A is called paracompact (resp.*-paracompact) if A is a α-paracompact (resp.α*-paracompact) for every $\alpha \in M(L)$. (L^X, δ) is paracompact (resp.*-paracompact) if $\underline{1}$ is paracompact (resp.*-paracompact). Where a collection A refines B (A B) if for every $A \in A$, $\exists\, B \in B$ such that $A \leq B$.

2.13 Definition Let (L^X, δ) be an *L-Fts*. $A = \{A_t : t \in T\} \subset L^X$, $x_\lambda \in M(L^X)$. A is called point finite at x_λ if $x_\lambda \prec A_t$ for at most finitely many $t \in T$. And A is *-point finite at x_λ if there exists at most finitely many $t \in T$ such that $x_\lambda \prec \chi_{At_{(0)}}$, where $At_{(0)} = \{ x \in X : A_t(x)\ 0 \}$. A is called point finite (resp. *-point finite) for short , if A is point finite (resp. *-point finite) at every molecule x_λ of L^X.

2.14 Definition Let (L^X, δ) be an *L-Fts*. $A \in L^X$, $\alpha \in M(L)$. A is called α-metacompact (resp. α*-metacompact) if every open α-Q-cover of A has a point finite (resp. *-point finite) open refinement which is also an α-Q-cover of A . A is called metacompact (resp.*-metacompact) if A is α-metacompact (resp. α*-metacompact)for every $\alpha \in M(L)$. And (L^X, δ) is metacompact (resp.*-metacompact) if $\underline{1}$ is metacompact (resp.*-metacompact).

3. A Characterization of Metacompactness

3.1 Definition [4] Let (L^X, δ) be an *L-Fts*. $A = \{A_t : t \in T\} \subset L^X$ is a closure preserving collection if for every subfamily A_0 of A, cl $[\vee A_0] = \vee$ cl A_0.

3.2 Proposition A point finite closure preserving closed collection is always locally finite.

3.3 Remark

(i) A collection $U = \{U : U \in U\}$ is locally finite implies that so is $\{cl\ U : U \in U\}$.

(ii) Similar to the Proposition 3.2 it can be shown that a *-point finite closure preserving collection is always *-locally finite.

3.4 Proposition Let (L^X, δ) be an *L-Fts*. $\alpha \in M(L)$, $A \in L^X$, $B \in \delta'$. Then

(i) If A is α-metacompact then so is $A \wedge B$

(ii) If A is metacompact then so is $A \wedge B$

3.5 Remark A result similar to that of Proposition 3.4 can be obtained for α*-metacompact and *-metacompact spaces also.

From the proposition 3.4 and remark 3.5 it follows clearly that

3.6 Theorem α-metacompactness , α*-metacompactness, α-metacompactness and *-metacompactness are all closed hereditary.

3.7 Theorem Let (L^X,δ) be a weakly induced *L-Fts*. Then the following conditions are equivalent.

(i) (L^X,δ) is metacompact.

(ii) There exists $\alpha \in M(L)$ such that (L^X,δ) is α-metacompact.

(iii) (X, $[\delta]$) is metacompact

3.8 Theorem Let (L^X,δ) be a weakly induced *L-Fts*. Then the following conditions are equivalent.

(i) (L^X,δ) is *-metacompact.

(ii) There exists $\alpha \in M(L)$ such that (L^X,δ) α*-metacompact.

(iii) (X, $[\delta]$) is metacompact

3.9 Theorem If (L^X,δ) is a weakly induced *L-Fts*, then the following are equivalent

(i) (L^X,δ) is metacompact.

(ii) For every $\alpha \in M(L)$, every well monotone open α-Q-cover of $\underline{1}$ has a point finite open refinement which is also an α-Q-cover of $\underline{1}$.

(iii) There exists an $\alpha \in M(L)$ such that every well monotone open α-Q-cover of $\underline{1}$ has a point finite open refinement which is also an α-Q-cover of $\underline{1}$.

3.10 Lemma Let (L^X,δ) be a weakly induced *L-Fts* and $\alpha \in M(L)$. Then if every directed open α-Q-cover of $\underline{1}$ has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$ then (L^X,δ) is metacompact.

3.11 Lemma Let (L^X,δ) be a weakly induced metacompact *L-Fts* and $\alpha \in M(L)$. Then every directed open α-Q-cover of $\underline{1}$ has a closure preserving closed refinement which is also an open α-Q-cover of $\underline{1}$.

3.12 Lemma Let (L^X,δ) be an *L-Fts* and $\alpha \in M(L)$. Then the following are equivalent.

(i) Every directed open α-Q-cover of $\underline{1}$ has a closure preserving closed refinement which is also an open α-Q-cover of $\underline{1}$.

(ii) For every α-Q-cover U of $\underline{1}$, U^F has a closure preserving closed refinement which is also an open α-Q-cover of $\underline{1}$. Where U^F is the collection of all unions of finite sub collections from U.

Combining Theorem 3.7, 3.9 , Lemmas 3.10, 3.11, and 3.12 , we get the following characterization of metacompactness in a weakly induced *L-Fts*.

3.13 Theorem Let (L^X,δ) be a weakly induced *L-Fts* . Then the following are equivalent

(i) (L^X,δ) is metacompact.

(ii) There exists $\alpha \in M(L)$ such that (L^X,δ) α-metacompact.

(iii) (X, [δ]) is metacompact

(iv) For every $\alpha \in M(L)$, every well monotone open α-Q-cover of $\underline{1}$ has a point finite open refinement which is also an α-Q-cover of $\underline{1}$.

(v) There exists an $\alpha \in M(L)$ such that every well monotone open α-Q-cover of $\underline{1}$ has a point finite open refinement which is also an α-Q-cover of $\underline{1}$.

(vi) For every $\alpha \in M(L)$, every directed open α-Q-cover of $\underline{1}$ has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$.

(vii) There exists an $\alpha \in M(L)$ such that every directed open α-Q-cover of $\underline{1}$ has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$.

(viii) For every $\alpha \in M(L)$, every open α-Q-cover U of $\underline{1}$, U^F has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$.

(ix) There exists an $\alpha \in M(L)$ such that for every open α-Q-cover U of $\underline{1}$, U^F has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$.

4. Invariant Theorems

4.1 Result If (L^X,δ) and (L^Y,μ) are two weakly induced *L-Fts,* then

(i) If the map $f^{\rightarrow}: L^X \rightarrow L^Y$ is L-fuzzy continuous , then f: (X, [δ]) $\rightarrow$ (X, [μ]) is continuous.

(ii) If the map $f^{\rightarrow}: L^X \rightarrow L^Y$ is L-fuzzy closed , then f: (X, [δ]) $\rightarrow$ (X, [μ]) is closed.

(iii) If the map $f^{\rightarrow}: L^X \rightarrow L^Y$ is L-fuzzy open , then f: (X, [δ]) $\rightarrow$ (X, [μ]) is open.

4.2 Theorem If (L^X,δ) and (L^Y,μ) are two weakly induced *L-Fts* . Then if $f^{\rightarrow}: L^X \rightarrow L^Y$ is perfect, then so is f: (X, [δ]) $\rightarrow$ (X, [μ]).

Proof

Let $y_\alpha \in L^Y$. Since $f^{\rightarrow}: L^X \rightarrow L^Y$ is perfect, $f^{\leftarrow}(y_\alpha)$ is N-compact. Now to prove f: (X, $[\delta]$) $\rightarrow$ (X, $[\mu]$) is enough to prove that $f^{\leftarrow}(y)$ is compact for every $y \in Y$. Now let $\mathbf{U} \subset [\delta]$ be an open cover of $f^{\leftarrow}(y)$. Consider $\mathbf{U}^* = \{\chi_u: U \in \mathbf{U}\}$. This is clearly an open α-Q-cover of $f^{\leftarrow}(y_\alpha)$. For, let $x_\alpha \leq f^{\leftarrow}(y_\alpha)$. i.e, $f^{\leftarrow}(y_\alpha)(x) = y_\alpha(f(x)) \geq \alpha$. Now let $U \in \mathbf{U}$ be such that $x \in U$. This is possible since $\mathbf{U}$ is a cover of $f^{\leftarrow}(y)$. Then $\chi_u(x) \geq y_\alpha(x) \geq \alpha$. i.e., $\chi_u(x) \geq \alpha$ or $x_\alpha \leq \chi_u$. Hence clearly $x_\alpha q \chi_u$. Hence $\{\chi_u: U \in \mathbf{U}\}$ is an open α-Q-cover of $f^{\leftarrow}(y_\alpha)$.

Again $f^{\leftarrow}(y_\alpha)$ being N-compact, there exists a finite sub collection $\mathbf{U}_s^*$ of $\mathbf{U}^*$ which is also an $\bar{\alpha}$-Q-cover of $f^{\leftarrow}(y_\alpha)$. Let $\mathbf{U}_s^* = \{\chi_{u1}, \chi_{u2}, \chi_{u3}, \ldots\ldots, \chi_{uk}\}$. Then clearly $\{U_1, U_2, \ldots\ldots\ldots U_k\}$ will be a finite sub cover of $f^{-1}(y)$. This completes the proof.

Since para(*-para), meta (*-meta) compactnesses are good extensions of para (meta) compactness respectively, we readily have,

4.3 Theorem If (L^X,δ) and (L^Y,μ) are two weakly induced *L-Fts* and $f^{\rightarrow}: L^X \rightarrow L^Y$ be a perfect map. Then (L^X,δ) is para(*-para) if and only if (L^Y,μ) is para(*-para).

4.4 Theorem If (L^X,δ) and (L^Y,μ) are two weakly induced *L-Fts* and $f^{\rightarrow}: L^X \rightarrow L^Y$ be a perfect map. Then (L^X,δ) is meta(*-meta) if and only if (L^Y,μ) is meta(*-meta).

4.5 Remark We remark that for the case of meta (*-meta) compact spaces, we can even relax the restriction on fibers. That is, the condition $f^{\leftarrow}(y)$ is *N*-compact can be relaxed. In [7] author has proved this result in [0,1] fuzzy context.

Now we give the analogues result in the L-fuzzy context.

4.6 Theorem Let (L^X,δ) and (L^Y,μ) are two weakly induced *L-Fts* and $f^{\rightarrow}: L^X \rightarrow L^Y$ be continuous and closed. Then if (L^X,δ) is meta (*-meta) then so is (L^Y,μ).

Proof

Let $\boldsymbol{U} \subset \mu$ be an open α-Q-cover of $\underline{1}$. Now by Theorem 3.13 (viii), it is enough to prove that $\boldsymbol{U}^F$ has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$. Given that $f^{\rightarrow}$ is continuous. Therefore $f^{-1}(U) \in \delta$ for any $U \in \mathbf{U}$. Let $\mathbf{W} = \{f^{-1}(U)$ $U \in \mathbf{U}\}$ be an open α-Q-cover of $\underline{1}$. Now since $\underline{1}$ is metacompact, it follows that $\boldsymbol{W}^F$ has a closure preserving α-Q-cover refinement say $\boldsymbol{F}$ by closed fuzzy sets. Since $f^{\rightarrow}$ is closed, it follows that f(F) is closed for every F∈F . Now $\{f(F): F\in F\}$ is the required closure preserving closed α-Q-cover of U^F.

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