Characterizations of prime fuzzy ideals of a semigroup [1]

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Abstract: The aim of this paper is to study prime fuzzy ideals of a semigroup S, to characerize prime fuzzy ideals of S by fuzzy points and fuzzy left (right) ideals of S.

Keywords: Fuzzy semigroup; Fuzzy ideal; prime fuzzy ideal; fuzzy point.

In [3], Ahsan, among others, has introduced the notion of prime fuzzy ideals of a semigroup S, and proved that a semigroup S is semisimple if and only if every fuzzy ideal of S is the intersection of all prime fuzzy ideals of S containing it. As a continuation of the results of [3], in this paper we give a characterization of prime fuzzy ideals of a semigroup S.

Throughout this paper, S always denote a semigroup. A function f from S to the unit interval [0, 1] is a *fuzzy subset* of S. Let f and g be two fuzzy subsets of S. The *product* $f \circ g$ is defined by

$$(f \circ g)(x) = \begin{cases} \sup\limits_{x=yz}[\min\{f(y), f(z)\}]; \\ 0, \quad \text{if } x \text{ is not expressible as } x = yz \text{ [3].} \end{cases}$$

As is well known [4], this operation $\circ$ is associative.

For any fuzzy subset f of S, it is obvious that $f = \bigcup\limits_{y_s \in f} y_s$, f is called a *fuzzy left ideal* of S if $S \circ f \subseteq f$, and f is called a *fuzzy right ideal* of S if $f \circ S \subseteq f$. A fuzzy subset f of S is called a *fuzzy ideal* if f is both a fuzzy left ideal and a fuzzy right ideal of S. A fuzzy ideal f of S is called *prime* if for any fuzzy ideals g and h, $g \circ h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$ [3]. By Definition of product of two fuzzy subsets of S, we have

Lemma 1 *Let f, g and h be fuzzy subsets of S. Then*

$$f \circ (g \cup h) \subseteq f \circ g \cup f \circ h.$$

Lemma 2 [2,5] *Let a_λ be a fuzzy point of S. Then the fuzzy ideal generated by a_λ, denoted by (a_λ), is*

$$(\forall x \in S) \quad (a_\lambda)(x) = \begin{cases} \lambda, & x \in (a); \\ 0, & \textit{otherwise}, \end{cases}$$

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where (a) *is an ideal of* S *generated by* a.

The statements 1), 3) and 4) of the following lemma are due to Theorems 2.5 and 2.6 in [5] or it is easy to check by these Theorems.

Lemma 3 *Let* a_λ *be a fuzzy point of* S. *The following are true:*

1)$(\forall x \in S)$ $(S \circ a_\lambda \circ S)(x) = \begin{cases} \lambda, & x \in SaS; \\ 0, & otherwise. \end{cases}$

2)$(\forall x \in S)$ $(a_\lambda \circ S \circ b_\mu)(x) = \begin{cases} \lambda \wedge \mu, & x \in aSb; \\ 0, & otherwise. \end{cases}$

3) $a_\lambda \circ b_\mu = (ab)_{\lambda \wedge \mu}$ *for all fuzzy points* a_λ *and* b_μ *of* S.

4) $(a_\lambda) = a_\lambda \cup a_\lambda \circ S \cup S \circ a_\lambda \cup S \circ a_\lambda \circ S$.

5) $(a_\lambda)^3 \subseteq S \circ a_\lambda \circ S$.

Lemma 4 *For any* $x, y \in S$, *the following statement is true:*

$$(x)(y) \subseteq xy \cup xyS \cup xSy \cup Sxy \cup SxyS \cup SxSy \cup SxSyS.$$

Lemma 5 *Let* f *and* g *be fuzzy subsets of* S , *and* $x_r \in f$, $y_s \in g$.

1) *If* f *and* g *are fuzzy right ideals of* S, *then* $(x_r) \circ (y_s) \subseteq f \circ g \cup S \circ f \circ g$.

2) *If* f *and* g *are fuzzy left ideals of* S, *then* $(x_r) \circ (y_s) \subseteq f \circ g \cup f \circ g \circ S$.

Proof Suppose for any $x_r, y_s \in S$, by Definition of operation "$\circ$" and Lemma 2, we have

$$(x_r) \circ (y_s)(w) = \begin{cases} r \wedge s, & w \in (x)(y); \\ 0, & \text{otherwise}, \end{cases}$$

where (x), (y) is an ideal of S generated by x and y respectively.
On the other hand,

$$\begin{aligned}(f \circ g \cup S \circ f \circ g)(w) &= f \circ g(w) \vee S \circ f \circ g(w) \\ &= \bigvee_{w=zt} (f(z) \wedge g(t)) \vee \bigvee_{w=suv} (S(s) \wedge f(u) \wedge g(v)).\end{aligned}$$

Then

$$(f \circ g \cup S \circ f \circ g)(w) \geq \bigvee_{w=zt} (x_r(z) \wedge y_s(t)) \vee \bigvee_{w=suv} (S(s) \wedge x_r(u) \wedge y_s(v)).$$

For any $w \in (x)(y)$, by Lemma 4, there exist $s_1, s_2, \cdots, s_{10}$ such that $w = xy$, or $w = xys_1$ or $w = xs_2y$, or $w = s_3xy$, or $w = s_4xys_5$, or $w = s_6xs_7y$, or $w = s_8xs_9ys_{10}$.
A) If $w = xy$, then

$$(f \circ g \cup S \circ f \circ g)(w) \geq x_r(x) \wedge y_s(y) = r \wedge s.$$

B) If $w = xys_1$, then

$$\begin{aligned}(f \circ g \cup S \circ f \circ g)(w) &\geq f(x) \wedge g(ys_1) \\ &\geq f(x) \wedge g(y) \ \ \text{(since } g \text{ is a fuzzy right ideal)} \\ &\geq x_r(x) \wedge y_s(y) = r \wedge s.\end{aligned}$$

By the same way, if $w = xs_2y$, we can prove that

$$f \circ g \cup S \circ f \circ g)(w) \geq r \wedge s.$$

C) If $w = s_3xy$, then

$$\begin{aligned}(f \circ g \cup S \circ f \circ g)(w) &\geq S(s_3) \wedge f(x) \wedge g(y) \\ &\geq x_r(x) \wedge y_s(y) = r \wedge s.\end{aligned}$$

D) If $w = s_4xys_5$, then

$$\begin{aligned}(f \circ g \cup S \circ f \circ g)(w) &\geq S(s_4) \wedge f(x) \wedge g(ys_5) \geq S(s_4) \wedge f(x) \wedge g(y) \\ &\geq x_r(x) \wedge y_s(y) = r \wedge s.\end{aligned}$$

Similar to D), we can prove that

$$(f \circ g \cup S \circ f \circ g)(w) \geq r \wedge s$$

holds in other cases of w.

By the proof above, we have for any $w \in S$,

$$(f \circ g \cup S \circ f \circ g)(w) \geq (x_r) \circ (y_s)(w).$$

The proof of 2) is similar to 1), we omit.

The following theorem is what we desire.

Theorem *Let f be a fuzzy ideal of S. Then the following statements are equivalent:*

1) *f is a prime fuzzy ideal of S;*
2) *For any $x_r, y_s \in S$, if $x_r \circ S \circ y_s \subseteq f$, then $x_r \in f$, or $y_s \in f$;*
3) *For any $x_r, y_s \in S$, if $(x_r) \circ (y_s) \subseteq f$, then $x_r \in f$, or $y_s \in f$;*
4) *If g and h are fuzzy right ideals of S such that $g \circ h \subseteq f$, then $g \subseteq f$, or $h \subseteq f$;*
5) *If g and h are fuzzy left ideals of S such that $g \circ h \subseteq f$, then $g \subseteq f$, or $h \subseteq f$;*
6) *If g is a fuzzy right ideal of S and h is a left ideal of S such that $g \circ h \subseteq f$, then $g \subseteq f$, or $h \subseteq f$.*

Proof 1)$\Longrightarrow$ 2). For any $x_r, y_s \in S$, if $x_r \circ S \circ y_s \subseteq f$. Then

$$(S \circ x_r \circ S) \circ (S \circ y_s \circ S) \subseteq S \circ (x_r \circ S \circ y_s) \circ S \subseteq f.$$

Since $S \circ x_r \circ S$ and $S \circ y_s \circ S$ are fuzzy ideal of S. Then we have $S \circ x_r \circ S \subseteq f$ or $S \circ y_s \circ S \subseteq f$. Say $S \circ x_r \circ S \in f$, then by Lemma 3(5),

$$(x_r)^3 \subseteq S \circ x_r \circ S \subseteq f.$$

Since f is prime, we have $x_r \in (x_r) \subseteq f$.

2)$\Longrightarrow$ 3). Let $(x_r) \circ (y_s) \subseteq f$. By Lemma 1 and Lemma 3(4), then

$$\begin{aligned}x_r \circ S \circ y_s &\subseteq x_r \circ (y_s \cup y_s \circ S \cup S \circ y_s \cup S \circ y_s \circ S) \\ &\subseteq (x_r) \circ (y_s) \subseteq f.\end{aligned}$$

By 2), thus $x_r \in f$, or $y_s \in f$.

3)$\Longrightarrow$ 4). Let g and h be fuzzy right ideals of S such that $g \circ h \subseteq f$. If $g \not\subseteq f$. Then there exists a $x \in S$ such that $g(x) \not\leq f(x)$. So $x_{g(x)} \notin f$. For any $y_s \in h$, by Lemma 5, then

$$(x_{g(x)}) \circ (y_s) \subseteq g \circ h \cup S \circ g \circ h \subseteq f \cup S \circ f \subseteq f.$$

By assumption, $y_s \in f$. Thus $h = \bigcup\limits_{y_s \in h} y_s \in f$.

3)$\Longrightarrow$ 5). Similar to the proof of 3)$\Longrightarrow$ 4).

5)$\Longrightarrow$ 1), 4)$\Longrightarrow$ 1) and 6)$\Longrightarrow$ 1) are obvious.

3)$\Longrightarrow$ 6). Let g be a fuzzy right ideal of S and h a fuzzy left ideal of S such that $g \circ h \subseteq$. If $g \not\subseteq f$. Then there exists a fuzzy point $x_r \in h$ such that $x_r \notin f$. For any $y_s \in h$,

$$\begin{aligned}
(x_r) \circ (y_s) &= (x_r \cup S \circ x_r \cup x_r \circ S \cup S \circ x_r \circ S) \circ (y_s) \\
&\subseteq (g \cup S \circ g) \circ (y_s \cup y_s \circ S \cup S \circ y_s \cup S \circ y_s \circ S) \\
&\subseteq (g \cup S \circ g) \circ (h \cup h \circ S) \\
&\subseteq g \circ h \cup g \circ h \circ S \cup S \circ g \circ h \cup S \circ g \circ h \circ S.
\end{aligned}$$

Since $g \circ h \subseteq f$, so we have $(x_r) \circ (y_s) \subseteq f$. Thus by hypothesis, $y_s \in f$. Therefore $h = \bigcup\limits_{y_s \in h} y_s \in f$.

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