A New Expression of Extention Principle

Guo Si-zong, Wang Lei
Department of Basic Science Liaoning Technical University
FuXin Liaoning 123000,China

Abstract: This paper firstly presents a decomposition theorem about fuzzy sets, which is based on "Fuzzy Point". Then a new expression of fuzzy extention principle is introduced which is based on the decomposition theorem. The new extention principle and the old one are coincident, but the new one is simpler and easier to understand. Morever, we can get the reasonable explaination of fuzzy compound function and the composite operation on multivariate fuzzy relation by the present extention principle.
Key Words: fuzzy set; fuzzy mapping; extension principle;decomposition theorem

1 Introduction

Extention principle is the basic principle among fuzzy set theories, it gives the structured description of the fuzzy subset A (image $f(A)$) in universe X on the condition that the mapping f is an ordinary mapping of X into Y. Fuzzy extention principle was first put forward by Zadeh[1], and many people have done a lot of discussion on its different equivalent forms[2]-[6]. Fuzzy mapping f is the mapping of X into fuzzy set class $\boldsymbol{F}$(Y). This concept is widespread and important in both fuzzy mathematics theories and application of practical projects. Such as interval value function, fuzzy value function, differentiation of fuzzy value function, composite operation on fuzzy relation and so on, they are all special examples of fuzzy mapping. Therefore, it is important for consummating fuzzy mapping theory and studying fuzzy relation operation to establish extension principle on the basis of fuzzy mapping. The present expression of extension principle is the union of λ—level set of fuzzy set. In this paper, we put forward a new expression of extension principle, which is equal to the traditional extension principle. this new expression abandons the form of λ—level set, so that the extension principle becomes simpler and easier to understand. Meanwhile, we also depict the expression of fuzzy compound mapping and the composite operation on multivariate fuzzy relation by this new expression of extension principle.

2 Basic conception

Set $\boldsymbol{P}$(X) and $\boldsymbol{F}$(X) denote the ensemble of ordinary sets and fuzzy sets over universe X respectively. Let A be a fuzzy set of universe X, where $A \in \boldsymbol{F}$ (X), its membership function is $\mu_A(x)$, for $\forall \lambda \in$ (0, 1), we call the ordinary set

/

$$A_\lambda = \{x \mid \mu_A(x) \ge \lambda, x \in X\} \tag{1}$$

the λ—level set of A in X

Let $A \in \boldsymbol{P}$ (X), for $\forall \lambda \in (0, 1)$, we define the fuzzy set $\lambda * A$ in X and the membership function as follows

$$\mu_{\lambda * A}(x) = \begin{cases} \lambda, x \in A \\ 0, x \notin A \end{cases} \tag{2}$$

For $\forall A \in \boldsymbol{F}$ (X), there is the decomposition theorem[1] :

$$A = \bigcup_{\lambda \in (0,1]} \lambda * A_\lambda \tag{3}$$

Suppose that f is a general mapping of X into Y, by f we can obtain the mapping of $\boldsymbol{P}$ (X) into $\boldsymbol{P}$ (Y) as following

$$f: \boldsymbol{P}(X) \to \boldsymbol{P}(Y)$$

$$A \to f(A) = \{ y \mid \exists\, x \in A, \text{and } y = f(x) \}$$

From above definition, we can easily get the following property :

$$\chi_{f(A)}(y) = \bigvee_{y=f(x)} \chi_A(x) \tag{4}$$

Where χ_A is the characteristic function of set A. Applying formula (4) and by the mapping f: X → Y we can obtain the following mapping:

$$f : \boldsymbol{F}(X) \to \boldsymbol{F}(Y)$$

$$A \to f(A) \in \boldsymbol{F}(Y)$$

the membership function of $f(A)$ is shown as following:

$$\mu_{f(A)}(y) = \bigvee_{y=f(x)} \mu_A(x) \qquad \forall\, y \in Y \tag{5}$$

Here, we don't intent to discuss the conclusion of the inverse mapping f^{-1}, for detailed discussion you may see the document[2].

3 New expression of fuzzy extension principle

Let X and Y be two universes , the mapping $\tilde{f}$: X $\to \boldsymbol{F}$ (Y) is called the fuzzy mapping of X into Y, If X and Y are all real number sets, and set N(Y) is the ensemble of bounded closed fuzzy numbers in Y, then the mapping $\tilde{f}$: X $\to N$ (Y) is called fuzzy value function.

Consider a special case, we choose the single point set $A = \{x\}$ in X , hence $\lambda_x = \lambda * \{x\}$ is called the fuzzy point of X, where $\lambda \in [0, 1]$.

From formula (2) , we have

$$\mu_{\lambda_x}(y) = \mu_{\lambda * \{x\}}(y) = \begin{cases} \lambda, y = x \\ 0, y \ne x \end{cases} \qquad \forall\, y \in Y \tag{6}$$

Let $A \in \boldsymbol{F}$ (X), for the given point $x \in$ X, the degree of membership with regard to fuzzy set A is $\mu_A(x)$.Consider the fuzzy point $\lambda_x = \lambda * \{x\}$ over x, and denote $\mu_A(x)$ as λ, hence it is called the fuzzy point which is restricted to fuzzy set A. In fact, $\lambda_x = A \cap \{x\}$ .From formula (6) we can easily get $\mu_{\lambda*\{x\}}(x) = \lambda$, so we have

$$\mu_{\mu_A(x)*\{x\}}(x) = \mu_A(x) \tag{7}$$

In fuzzy set theory, the traditional decomposition theorem decompounds fuzzy set as the union of λ—level set of the fuzzy set, However, fuzzy set can be expressed by the union of fuzzy points, which is restricted to the fuzzy set A in X. Therefore, we have the following decomposition theorem.

Theorem1 (decomposition theorem) If let $A \in \boldsymbol{F}$ (X) and its membership function is $\mu_A(x)$, then we have

$$A = \bigcup_{x \in X} \mu_A(x) * \{x\} \tag{8}$$

Proof: From (7), we may immediately obtain the proof.

By theorem 1, we can also give a new expression of fuzzy extention principle

Theorem 2 (extention principle) Consider the general mapping f: X → Y, by this mapping f we can obtain the following mapping:

$$f: \boldsymbol{P}(X) \to \boldsymbol{P}(Y)$$
$$A \to f(A) = \{ y \mid \exists\, x \in A, \text{ and } y = f(x) \}$$
$$= \bigcup_{x \in X} \chi_A(x) * \{f(x)\} \tag{9}$$

Proof: By expression (9) and the definition of the union operation of the set's characteristic function , we have

$$\chi_{f(A)}(y) = \bigvee_{x \in X} \chi_{\chi_A(x)*\{f(x)\}}(y) \tag{10}$$

By definition (2) and the expression (10), we have

$$\chi_{\chi_A(x)*\{f(x)\}}(y) = \begin{cases} \chi_A(x), & y = f(x) \\ 0, & y \neq f(x) \end{cases}$$

Above expression is Substituted in (10), we can obtain

$$\chi_{f(A)}(y) = \bigvee_{y=f(x)} \chi_A(x)$$

Obviously, the expressions (9) and (4) are equivalent..

Furthermore, by this mapping f:X → Y we can obtain the following mapping:

$$f : \boldsymbol{F}(X) \to \boldsymbol{F}(Y)$$
$$A \to f(A) = \bigcup_{x \in X} \mu_A(x) * \{f(x)\} \tag{11}$$

According to the proof method of theorem 2, the membership function of fuzzy set $f(A) \in \boldsymbol{F}$ (y) depicted by (11) is as follows:

$$\mu_{f(A)}(y) = \bigvee_{y=f(x)} \mu_A(x)$$

This is equal to (5) .

Definition 1 (extension principle of fuzzy mapping) Suppose that $\tilde{f}$: X → $\boldsymbol{F}$ (Y) is a fuzzy mapping , by this mapping $\tilde{f}$ we can obtain the following mapping:

$$\tilde{f}: \boldsymbol{F}(X) \to \boldsymbol{F}(Y)$$

$$A \to \tilde{f}(A) = \bigcup_{x \in X} \mu_A(x) * \tilde{f}(x) \qquad (12)$$

Where ,for any given point $x \in X$,

$$\mu_{\mu_A(x)*\tilde{f}(x)}(y) = \mu_A(x) \wedge \mu_{\tilde{f}(x)}(y)$$

Hence we have

$$\mu_{\tilde{f}(A)}(y) = \bigvee_{x \in X} [\mu_A(x) \wedge \mu_{\tilde{f}(x)}(y)] \qquad \forall\ y \in Y \qquad (13)$$

Definition 2 (multivariate extension principle) Let $A_i \in \boldsymbol{F}(X_i)$, $i=1, 2, \cdots, n$. Consider the n-ary fuzzy mapping $\tilde{f}: X_1 \times X_2 \times \cdots \times X_n \to \boldsymbol{F}(y)$, by this mapping $\tilde{f}$ we can obtain the following mapping:

$$\tilde{f}: \boldsymbol{F}(X_1) \times \boldsymbol{F}(X_2) \times \cdots \times \boldsymbol{F}(X_n) \to \boldsymbol{F}(Y)$$

$$(A_1, A_2, \cdots, A_n) \to \tilde{f}(A_1, A_2, \cdots, A_n) \in \boldsymbol{F}(Y)$$

We have

$$\tilde{f}(A_1, A_2, \cdots, A_n) = \bigcup_{x_1 \in X_1} \cdots \bigcup_{x_n \in X_n} [\mu_{A_1 \times A_2 \times \cdots \times A_n}(x_1, x_2, \cdots, x_n) * \tilde{f}(x_1, x_2, \cdots, x_n)]$$

The membership function is as follows:

$$\mu_{\tilde{f}(A_1, A_2, \cdots, A_n)}(z) = \bigvee_{x_1 \in X_1} \cdots \bigvee_{x_n \in X_n} [\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge \cdots \wedge \mu_{A_n}(x_n) \wedge \mu_{\tilde{f}(x_1, x_2, \cdots, x_n)}(z)] \qquad (14)$$

4 Application of extension principle

4-1 Fuzzy relation and fuzzy transformation

First, we discuss the simpler binary fuzzy relation. fuzzy subset $R \in \boldsymbol{F}(X \times Y)$ in the Decare product set $X \times Y$ is called fuzzy relation of X into Y, for $(x, y) \in X \times Y$, the membership function $\mu_R(x,y) \in [0, 1]$ of fuzzy subset R is called the correlation degree which x and y satisfied the relation R.

Theorem 3 [2] If a given fuzzy mapping $\tilde{f}$ of X into Y , fuzzy relation R_f can be uniquely determined, where $R_f \mid_x = \tilde{f}(x)$ .For any given point $x \in X$ it has

$$\mu_{R_f|x}(y) = \mu_{\tilde{f}(x)}(y) = \mu_R(x,y) \qquad (15)$$

Conversely, if the fuzzy relation $R \in \boldsymbol{F}(X \times Y)$ is given, then fuzzy mapping $\tilde{f}_R$ of X into Y can be uniquely determined, where $\tilde{f}_R(x) = R \mid x$.

Definition 3 Let $R \in \boldsymbol{F}(X \times Y)$, and

$$T_R: \boldsymbol{F}(X) \to \boldsymbol{F}(Y)$$

$$A \to T_R(A) = A \circ R$$

Then T_R is called fuzzy transformation which is induced by fuzzy relation R of X into Y .By definition of composition operation "∘" on fuzzy relation, we have

$$\mu_{T_R(A)}(y) = \mu_{A\circ R}(y) = \bigvee_{x\in X}[\mu_A(x)\wedge\mu_R(x,y)] \tag{16}$$

Theorem 4 Suppose that $\tilde{f}(x)$ is fuzzy mapping of X into Y determined by fuzzy relation $R\in \boldsymbol{F}(X\times Y)$, for any $A\in\boldsymbol{F}(X)$ it has

$$\tilde{f}(A) = A\circ R$$

Proof : From (15) and (16), the conclusion holds.

Theorem 5 Suppose that $\tilde{f}(x_1, x_2, \cdots, x_n)$ is fuzzy mapping of $X_1\times X_2\times\cdots\times X_n$ into Y determined by fuzzy relation $R\in \boldsymbol{F}(X_1\times X_2\times\cdots\times X_n\times Y)$, for $\forall(A_1, A_2, \cdots, A_n)\in\boldsymbol{F}(X_1)\times\boldsymbol{F}(X_2)\times\cdots\times\boldsymbol{F}(X_n)$ then we have

$$\tilde{f}(A_1, A_2, \cdots, A_n) = (A_1\times A_2\times\cdots\times A_n)\circ R$$

Proof: From (14), we have

$$\mu_{\tilde{f}(A_1,A_2,\cdots,A_n)}(y) = \bigvee_{x_1\in X_1}\cdots\bigvee_{x_n\in X_n}[\mu_{A_1}(x_1)\wedge\cdots\wedge\mu_{A_n}(x_n)\wedge\mu_{\tilde{f}(x_1,x_2,\cdots,x_n)}(y)]$$

And from

$$\mu_{(A_1\times A_2\times\cdots\times A_n)\circ R}(y) = \bigvee_{(x_1,x_2,\cdots,x_n)\in X_1\times X_2\times\cdots\times X_n}[\mu_{A_1\times A_2\times\cdots A_n}(x_1,x_2,\cdots,x_n)\wedge\mu_R(x_1,x_2,\cdots,x_n,y)]$$

Where

$$\mu_{A_1\times\cdots\times A_n}(x_1,\cdots,x_n) = \mu_{A_1}(x_1)\wedge\mu_{A_2}(x_2)\wedge\cdots\wedge\mu_{A_n}(x_n)$$

$$\mu_R(x_1,x_2,\cdots,x_n,y) = \mu_{\tilde{f}(x_1,x_2,\cdots,x_n)}(y)$$

So we can obtain

$$\mu_{(A_1\times A_2\times\cdots\times A_n)\circ R}(y) = \mu_{\tilde{f}(A_1,A_2,\cdots,A_n)}(y)$$

4-2 Fuzzy compound mapping

Suppose that $\tilde{f}: X\to\boldsymbol{F}(Y)$, $\tilde{g}: Y\to\boldsymbol{F}(Z)$ are two fuzzy mappings, by $\tilde{f}$ and $\tilde{g}$ we can determine the following mapping

$$\tilde{h}: X\to\boldsymbol{F}(Z)$$

$$x\to\tilde{h}(x) = \tilde{g}[\tilde{f}(x)] \tag{17}$$

Where $\tilde{h}$ is called the compound mapping of $\tilde{g}$ and $\tilde{f}$, denoted by $\tilde{g}\circ\tilde{f}$, .

Since $\forall x\in X$, and $\tilde{f}(x)\in\boldsymbol{F}(Y)$, and $\tilde{g}$ is mapping of Y into $\boldsymbol{F}(Z)$, then from (12) we have

$$\tilde{g}[\tilde{f}(x)] = \bigcup_{y\in Y}\mu_{\tilde{f}(x)}(y) * \tilde{g}(y) \qquad \forall x\in X \tag{18}$$

Its membership function of $\tilde{g}(\tilde{f})$ is as follows:

$$\mu_{\tilde{g}(\tilde{f})}(z) = \bigvee_{y\in Y}[\mu_{\tilde{f}(x)}(y)\wedge\mu_{\tilde{g}(y)}(z)] \qquad \forall x\in X \tag{19}$$

Then we consider the composition of fuzzy relation. Let $R\in\boldsymbol{F}(X\times Y)$, $S\in\boldsymbol{F}(Y\times Z)$, and Q is called the composition of fuzzy relation R and S, consequently, $Q\in\boldsymbol{F}(X\times Z)$ is fuzzy relation of X into Z, and

$$\mu_Q(x,z) = \bigvee_{y\in Y}[\mu_R(x,y) \wedge \mu_S(y,z)] \qquad \forall\ x\in X,\ z\in Z \tag{20}$$

Suppose that $\tilde{f}_R$ is fuzzy mapping of X into Y determined by fuzzy relation $R \in \boldsymbol{F}$ (X×Y), $\tilde{g}_S$ is fuzzy mapping of Y into Z determined by fuzzy relation S, then we have

$$\mu_{\tilde{f}_R(x)}(y) = \mu_R(x,y) \qquad \mu_{\tilde{g}_S(y)}(z) = \mu_S(y,z)$$

Put above result in (20) and relate the expression (19), we can get the following theorem.

Theorem 6 Suppose that $\tilde{f}$: X→$\boldsymbol{F}$ (Y), $\tilde{g}$: Y→$\boldsymbol{F}$ (Z) are two fuzzy mappings ,then fuzzy relation $Q_{\tilde{f}\circ\tilde{g}} \in \boldsymbol{F}$ (X×Z) can be uniquely determined by compound mapping $\tilde{g} \circ \tilde{f}$, where

$$Q_{\tilde{f}\circ\tilde{g}} = R_f \circ S_g \qquad R_f \mid_x = \tilde{f}(x) \quad S_g \mid_y = \tilde{g}\ (y)$$

Conversely, for given fuzzy relation $R \in \boldsymbol{F}$(X×Y) and $S \in \boldsymbol{F}$(Y×Z), the fuzzy mapping $\tilde{g} \circ \tilde{f}$: X→$\boldsymbol{F}$(Z) can be uniquely determined by the fuzzy compound relation $Q = R \circ S \in \boldsymbol{F}$ (X×Z) , where

$$\tilde{f}(x) = R \mid x \qquad \tilde{g}\ (y) = S \mid y$$

Reference

[1] Zadeh,L.A. Fuzzy Sets . Information and Control, 8(1965) , 338-353

[2] Luo Ceng-zong. Fuzzy Set Guide Theory (first volume). Beijing, The Press of Beijing Normal University, 1989.

[3] Zhai Jian-yin. General extension principle [J] Fuzzy System And Mathematic, 1995,9(1)

[4] Liu Hua-wen. A expression of max-min extension principle [J]. Lanzhou University Journal (Fuzzy System And Mathematic Special).(32)1996,,P:393-396.

[5] Jain,R. Tolerance analysis using fuzzy sets.Int.J.Syst.Sci.7(1976),No.12,1393-1401

[6] Dubois,D.,and Prade,H. Comment on "Tolerance analysis using fuzzy sets" and "A procedure for multiple aspect decision making." Int. J. Syst. Sci. 9(1978), No. 3,357-360