Difference Operation Defined over the Intuitionistic Fuzzy Sets

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Abstract:

An axiomatic definition of difference operation for intuitionistic fuzzy sets is given. The analytic expression of the difference operation is also given. At last, the properties of difference operation are discussed.

Keywords:

Intuitionistic fuzzy sets, Difference operation, Intuitionistic fuzzy generator.

Some operations are defined over the Intuitionistic Fuzzy Sets (IFSs) in [1-3]. Here we shall introduce a new one, and we shall show some of their basic properties.

Let a set E be fixed. An intuitionistic fuzzy set (IFS) A in E is an object of the following form (see [1]):

$$A = \{< x, \mu_A(x), \nu_A(x) > | \, x \in E\}$$

where the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$,

$$0 \le \mu_A(x) + \nu_A(x) \le 1$$

For two IFSs A and B, the following definitions are valid (see [1]):

$$A \subseteq B \quad \textit{iff} \quad (\forall x \in E)(\mu_A(x) \le \mu_B(x) \,\&\, \nu_A(x) \ge \nu_B(x))$$
$$A \supseteq B \quad \textit{iff} \quad B \subseteq A$$
$$A = B \quad \textit{iff} \quad (\forall x \in E)(\mu_A(x) = \mu_B(x) \,\&\, \nu_A(x) = \nu_B(x))$$
$$\overline{A} = \{< x, \nu_A(x), \mu_A(x) > | \, x \in E\}$$

Let us take the following set:

$$D = \{(x, y) \in [0,1] \times [0,1] \,|\, x + y \le 1\}$$

First we introduce operations on D as follows:

Definition 1. For every $(a_t, b_t) \in D, t \in T$, we define:

$$\vee_{t \in T}(a_t, b_t) = (\vee_{t \in T} a_t, \wedge_{t \in T} b_t);$$
$$\wedge_{t \in T}(a_t, b_t) = (\wedge_{t \in T} a_t, \vee_{t \in T} b_t);$$

$$(a_t, b_t)' = (b_t, a_t)$$

Definition 2. For each $(a_i, b_i) \in D, i = 1,2$. We define:

$$(a_1,b_1)=(a_2,b_2) \quad \mathit{iff} \quad a_1 = a_2 \,\&\, b_1 = b_2;$$
$$(a_1,b_1)\le(a_2,b_2) \quad \mathit{iff} \quad a_1 \le a_2 \,\&\, b_1 \ge b_2;$$
$$(a_1,b_1)<(a_2,b_2) \quad \mathit{iff} \quad (a_1,b_1)\le(a_2,b_2)\,\&\,(a_1,b_1)\ne(a_2,b_2).$$

It is easy to prove the following results.

Theorem 1. Let $\alpha, \alpha_t \in D, t \in T$, then

(1) $\alpha \wedge (\vee_{t\in T} \alpha_t) = \vee_{t\in T} (\alpha \wedge \alpha_t)$;

(2) $\alpha \vee (\wedge_{t\in T} \alpha_t) = \wedge_{t\in T} (\alpha \vee \alpha_t)$.

Theorem 2. The system $(D, \leq, \wedge, \vee)$ is a complete lattice with the order-reversing involution" ′ ". And it has maximal element $\tilde{1} = (1,0)$ and minimal element $\tilde{0} = (0,1)$.

Definition 3. (see [4]) An intuitionistic fuzzy complementation is a function Φ from D to D such that:

(1) $\Phi(\tilde{1}) = \tilde{0}$ *and* $\Phi(\tilde{0}) = \tilde{1}$;

(2) *For all* $x, y \in D$, *if* $x \le y$, *then* $\Phi(x) \ge \Phi(y)$.

For each intuitionistic fuzzy set A, that is $A = \{< x, \mu_A(x), \nu_A(x) > \mid x \in E\}$, we call

$$\Phi(A) = \{< x, \Phi(\mu_A(x), \nu_A(x)) > \mid x \in E\}$$

intuitionistic fuzzy complementary of the intuitionistic fuzzy set A.

Now, we give the axiomatic definition of difference operation for IFSs. In the following, we write $A(x) = (\mu_A(x), \nu_A(x))$ for IFS A and $x \in E$.

Definition 4. For two IFSs A and B, we define the difference operation of A and B as follows:

$$A - B = \{< x, f(A(x), B(x)) > \mid x \in E\}$$

where $f : D \times D \to D, (x, y) \mapsto f(x, y)$ satisfies the following axioms:

(1) $f(x,y) \le x$;

(2) $f(x,\tilde{0}) = x$;

(3) *If* $y \le z$, *then* $f(x,y) \ge f(x,z), f(y,x) \le f(z,x)$;

(4) $f(\tilde{1},x) = \Phi(x)$, where Φ is an intuitionistic fuzzy complementation.

In the following, we give the analytic expressions of the difference operation for IFSs.

Definition 5. For two IFSs A and B in E, we define the difference operation as follows:

$$A -_1 B = \{< x, \mu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \mu_B(x) > | \, x \in E\}$$

Obviously, for two IFSs A and B, $A -_1 B$ is an IFS. It is easy to prove the following results:

Theorem 3. Let A, B, C be IFSs in E, then we have

(1) $A -_1 B \subseteq A$;

(2) $A -_1 \phi = A$;

(3) *If* $B \subseteq C$, *then* $A -_1 B \supseteq A -_1 C$, $B -_1 A \subseteq C -_1 A$;

(4) $\overline{A} = X -_1 A$.

The above property (4) illustrates that the definition of difference operation and the definition of intuitionistic fuzzy complement given by Atanassov have identity. In fact, the intuitionistic fuzzy complement $\overline{A}$ is the special difference operation: $X -_1 A$.

Definition 6. (see [4]) A function $\varphi : [0,1] \to [0,1]$ will be called intuitionistic fuzzy generator if

$$\varphi(x) \leq 1 - x \text{ for all } x \in [0,1]$$

An intuitionistic fuzzy generator will be called continuous, decreasing and increasing if φ is continuous, decreasing and increasing, respectively.

Note that according to the definition above, $\varphi(0) \leq 1$ *and* $\varphi(1) = 0$.

Using the intuitionistic fuzzy generator φ, we can define the difference operation over the intuitionistic fuzzy sets as follows

Definition 7. Let $\varphi : [0,1] \to [0,1]$ be a decreasing intuitionistic fuzzy generator such that $\varphi(0)=1$, and A and B be two IFSs in E. We define

$$A -_2 B = \{< x, \mu_A(x) \wedge \varphi(1 - \nu_B(x)), \nu_A(x) \vee (1 - \varphi(\mu_B(x))) > | \, x \in E\}$$

Here, $A -_2 B$ is an intuitionistic fuzzy set. In fact, for all $x \in E$, $\mu_B(x) + \nu_B(x) \leq 1$, then $\mu_B(x) \leq 1 - \nu_B(x)$. Besides φ is decreasing, therefore $\varphi(\mu_B(x)) \geq \varphi(1 - \nu_B(x))$, then $0 \leq \varphi(1 - \nu_B(x)) + 1 - \varphi(\mu_A(x)) \leq 1$. So $A -_2 B$ is an intuitionistic fuzzy set.

Note that if we take $\varphi = N$ which is the standard negation N, N: $[0,1] \to [0,1]$ given by $N(x) = 1 - x$ for all $x \in [0,1]$, then $A -_2 B = A -_1 B$ holds. This means that operation $-_2$ is the generalization of operation $-_1$.

Definition 8. Let $\varphi : [0,1] \to [0,1]$ be a decreasing intuitionistic fuzzy generator such that $\varphi(0)=1$, and A and B be two IFSs in E. We define

$$A -_3 B = \{< x, \mu_A(x) \wedge \varphi(1-\nu_B(x)), \nu_A(x) \vee \varphi(1-\mu_B(x)) > | \, x \in E\}$$

Here, $A -_3 B$ is an intuitionistic fuzzy set.

In fact, for all $x \in E, \mu_B(x)+\nu_B(x) \leq 1$, then

$$\varphi(1-\nu_B(x))+\varphi(1-\mu_B(x)) \leq \nu_B(x)+\mu_B(x) \leq 1$$

Therefore, $A -_3 B$ is an intuitionistic fuzzy set.

Note that if we take $\varphi = N$ given by $N(x)=1-x$ for all $x \in [0,1]$, then we have $A -_3 B = A -_1 B$. This means the operation $-_1$ is a special case of operation $-_3$.

In the following, we give the properties of operation $-_2$ and operation $-_3$.

Theorem 4. Let A, B and C be intuitionistic fuzzy sets in E. We write the sign "$-$" to express the operation $-_2$ or operation $-_3$, then

(1) $A - B \subseteq A$;

(2) $A - \phi = A$;

(3) *If* $B \subseteq C$, *then* $A-B \supseteq A-C, B-A \subseteq C-A$;

(4) $X - A = \Phi(A)$. where $\Phi(A)$ is the intuitionistic fuzzy complementary of IFS A.

Proof. Only in the case $-_3$ we give the proof. The proof of case $-_2$ is similar.

The proofs of (1), (2) are straightforward.

(3) For all $x \in E$, from $\mu_B(x) \leq \mu_C(x), \nu_B(x) \geq \nu_C(x)$, i.e. $1-\mu_B(x) \geq 1-\mu_C(x), 1-\nu_B(x) \leq 1-\nu_C(x)$, considering φ is decreasing, we get

$$\varphi(1-\mu_B(x)) \leq \varphi(1-\mu_C(x)), \varphi(1-\nu_B(x)) \geq \varphi(1-\nu_C(x))$$

i.e.

$$\begin{aligned} &\mu_A(x) \wedge \varphi(1-\nu_B(x)) \geq \mu_A(x) \wedge \varphi(1-\nu_C(x)), \\ &\nu_A(x) \vee \varphi(1-\mu_B(x)) \leq \nu_A(x) \vee \varphi(1-\mu_C(x)) \end{aligned}$$

Therefore, from $B \subseteq C$ we have $A -_3 B \supseteq A -_3 C$.

The proof of the formula $B -_3 A \subseteq C -_3 A$ is similar to above. So we omit it.

(4) $X -_3 A = \{< x, 1 \wedge \varphi(1-\nu_A(x)), 0 \vee \varphi(1-\mu_A(x)) > | \, x \in E\}$
$= \{< x, \varphi(1-\nu_A(x)), \varphi(1-\mu_A(x)) > | \, x \in E\}$

We write $\Phi(\mu_A(x), \nu_A(x)) = (\varphi(1-\nu_A(x)), \varphi(1-\mu_A(x)))$. In the following, we only need to prove Φ is an intuitionistic fuzzy complementation.

(a) If $\mu_A(x) = 1$ *and* $\nu_A(x) = 0$, then $\Phi(1,0) = (\varphi(1-0), \varphi(1-1)) = (0,1) = \tilde{0}$.

If $\mu_A(x) = 0$ *and* $\nu_A(x) = 1$, then $\Phi(0,1) = (\varphi(1-1), \varphi(1-0)) = (1,0) = \tilde{1}$.

(b) Let A, B be two IFSs in E such that $A(x) \le B(x)$ for all $x \in E$, that is

$\mu_A(x) \le \mu_B(x)$ *and* $\nu_A(x) \ge \nu_B(x)$ *for all* $x \in E$. Then we have

$1-\nu_A(x) \le 1-\nu_B(x)$ *and* $1-\mu_A(x) \ge 1-\mu_B(x)$. Besides φ is decreasing, therefore

$\varphi(1-\nu_A(x)) \ge \varphi(1-\nu_B(x))$ *and* $\varphi(1-\mu_A(x)) \le \varphi(1-\mu_B(x))$, that is

$\Phi(\mu_A(x), \nu_A(x)) \ge \Phi(\mu_B(x), \nu_B(x))$ for all $x \in E$. □

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