[1]Fuzzy Congruence on Rings

Zhang Chengyi[1],Su Jinlin[2]

[1]Dep. of Math. And Comp.Sci., Hainan Normal University,Hainan,Haikou,571158,P.R.China.

[2]Dep. of Math., Northwest Minorities University Lanzhou,730030,P.R.China.

1. Introduction.

N.Kuroki[1,2,3] discussed the concept of a fuzzy congruence relation on a semigroup. F.A.AL-thukair discussed the fuzzy congruence pairs of inverse semigroups. It is natural to define the fuzzy congruence relation on a ring and study a ring R with the fuzzy congruence relation (abbr. FCR)

The purpose of this paper is to introduce the concept of fuzzy congruence, give some homomorphic properties of a ring with fuzzy congruence relation. We obtain that if R is a ring with FCR, then the kernel of FCR is a fuzzy ideal of R. We also prove some homomorphic theorems of a ring R with FCR. Moreover, some applications of the result of this paper are given.

2. Preliminaries.

Let R be a ring. A function α from $R\times R$ to the unit interval [0,1] is called a fuzzy relation on R. Let α and β be two fuzzy relations on S, the product $\alpha \circ \beta$ of α and β is defined by

$$(\alpha \circ \beta)(a,b) = \sup_{x\in R}[\min\{\alpha(a,x), \beta(x,b)\}]$$

for all $a,b \in R$,and $\alpha \le \beta$ is defined by $\alpha(x) \le \beta(x)$ for all $x \in R$.

A fuzzy set of a set X is a function μ from X to the unit interval [0,1], and all fuzzy sets of X denote by **F**(X) . As is well-known, $\mu_\lambda = \{x \in X \mid \mu(x) \ge \lambda\}$ is a λ-cut set of μ and $\mu_{(\lambda)} = \{x \in X \mid \mu(x) > \lambda\}$ is a strong λ-cut set of μ.

A fuzzy relation μ on R is called fuzzy equivalence relation on R if

(E.1) $\mu(a,a) = 1$ for all $a \in R$. (fuzzy reflexive)

(E.2) $\mu(a,b) = \mu(b,a)$ for all $a,b \in R$. (fuzzy symmetric)

(E.3) $\mu \circ \mu \subseteq \mu$. (fuzzy transitive).

Definition .2.1 If μ is a fuzzy equivalence relation on a ring R, then μ is called a fuzzy congruence relation on R if

$(c.1)$ $\mu(a+x, b+x) \ge \mu(a,b)$;

$(c.2)$ $\mu(ax,bx) \ge \mu(a,b)$ and $\mu(xa,xb) \ge \mu(a,b)$

[1] Project supported by the NSF of Hainan(10101)

for all $a,b \in R$.

We denote by χ_f the characteristic function of a binary relation f on R. We denote Con(R) and con $_f$ (R) the set of all congruence and the all fuzzy congruence on R respectively. Then we have the following conclusions.

Proposition 2.1 Let f is a binary relation on a ring R. Then f is an equivalence (a congruence) on R if and only if χ_f is a fuzzy equivalence (a fuzzy congruence) on R .

Proof: It is clear by the Theorem 2.4 in [3].

Let μ be a fuzzy equivalence relation R. For each $a \in R$, we define a fuzzy subset μ_a on R as follows:

$$\mu_a(x) = \mu(a,x) \text{ for all } x \in R.$$

Then we have the following.

Proposition 2.2 Let μ be a fuzzy congruence relation on R, and let θ be a zero element of R. Then μ_θ is a fuzzy ideal of R.

Proof: Let μ be a fuzzy congruence relation on R, and for all $x, y \in R$,

$$\mu_\theta(x-y) = \mu(\theta, x-y) = \mu(y-y, x-y) \geq \mu(y,x) = \mu(x,y)$$

$$\geq (\mu \circ \mu)(x,y) = \sup_{z \in R}[\min\{\mu(x,z), \mu(z,y)\}]$$

$$\geq \min\{\mu(x,\theta), \mu(\theta,y)\} = \min\{\mu_\theta(x), \mu_\theta(y)\}$$

and for all $r \in R$,

$$\mu_\theta(rx) = \mu(\theta, rx) = \mu(r\theta, rx) \geq \mu(\theta, x) = \mu_\theta(x)$$

Similarly, $\mu_\theta(rx) \geq \mu_\theta(r)$, then we have $\mu_\theta(rx) \geq \max\{\mu_\theta(r), \mu_\theta(x)\}$,Thus, μ_θ is a fuzzy ideal of R.

Proposition 2.3. Let μ be a fuzzy congruence relation on R, and $a \in R$.Then. for all $x \in R$, $\mu_a(x) = \mu_\theta(x-a)$.

Proof: we only prove $\mu\ (a,x) = \mu\ (\theta, x-a)$ for all $x \in R$.

$$\mu(\theta, x-a) = \mu(a-a, x-a)$$

$\geq \mu(a,x) \geq (\mu \circ \mu)(a,x)$

$= \sup_{y\in R}[\min\{\mu(a,y), \mu(y,x)\}]$

$\geq \min\{\mu(a,a), \mu(a,x)\}$

$= \mu(a,x) = \mu(a, x-a+a)$

$\geq \mu(\theta, x-a)$,

Then $\mu_a(x) = \mu_\theta(x-a)$ for all $x \in R$.

Proposition 2.4 Let μ be a fuzzy congruence relation on R , then for each $a,b \in R$, $\mu_a = \mu_b$ if and only if $\mu_\theta(a-b) = 1$.

Equivalent: $\mu_a = \mu_b$ if and only if $\mu\ (a,b) = 1$.

Proof: Suppose that $\mu_a = \mu_b$, then for all $x \in R$. $\mu(a,x) = \mu(b,x)$,it implies that $\mu_\theta(a-b) = \mu(a-b,\theta) = \mu(a,b) = \mu(b,a) = \mu(a,a) = 1$;

Conversely, if $\mu_\theta(a-b) = \mu(\theta, a-b) = 1$, then for all $x \in R$,

$\mu_a(x) = \mu(a,x) \geq \mu(\theta, x-a) \geq (\mu \circ \mu)(\theta, x-a)$

$= \sup_{y\in R}[\min\{\mu(\theta,y), \mu(y, x-a)\}]$

$\geq \min\{\mu(\theta, b-a), \mu(b-a, x-a)\} = \mu(b-a, x-a)$

$\geq \mu(b,x) = \mu_b(x)$.

And so $\mu_b \supseteq \mu_a$ .By symmetry, we have $\mu_a \subseteq \mu_b$, Thus we obtain that $\mu_a = \mu_b$.

Let μ be a fuzzy congruence relation on R. For all $a,b \in R$, the addition $\mu_a \oplus \mu_b$ and product $\mu_a \circ \mu_b$ of μ_a and μ_b are defined respectively by

$$\mu_a \oplus \mu_b(x) = \begin{cases} \sup_{x=y+z}[\min\{\mu_a(y), \mu_b(z)\}], & if\quad x = y+z, \\ 0 \quad if \quad x \quad is\ not\ expressible\ as & x = y+z, \end{cases}$$

and $$\mu_a \circ \mu_b(x) = \begin{cases} \sup_{x=yz}[\min\{\mu_a(y), \mu_b(z)\}], & if\quad x = yz \\ 0 & other\ wise. \end{cases}$$

Lemma2.5[Proposition in[]] Let μ_θ be a fuzzy additive subgroup of R , then for all

$y,z \in R$,if $\mu_\theta(y) \neq \mu_\theta(z)$,then $\mu_\theta(y+z) = \min\{\mu_\theta(y), \mu_\theta(z)\}$.

Proposition 2.6 Let μ be a fuzzy congruence relation on R. Then, $\mu_a \oplus \mu_b = \mu_{a+b}$.

Proof: First, we prove the binary operations $\oplus$ are well-defined. Assume that $\mu_a = \mu_b$ and $\mu_c = \mu_d$, then by Proposition 2.4, we have $\mu(a,b) = \mu(c,d) = 1$ .Thus

$$\mu(a+c,b+d) = \mu_\theta((a-b)+(c-d)) \geq \mu_\theta(a-b) \wedge \mu_\theta(c-d) = \mu(a,b) \wedge \mu(c,d) = 1$$

So, we have $\mu_a \oplus \mu_c = \mu_b \oplus \mu_d$.

For each $y,z \in R$ satisfies $x = y+z$.

$$\mu_{a+b}(x) = \mu(a+b,x) = \mu_\theta[x-(a+b)]$$

$$= \mu_\theta[y+z-(a+b)]$$

$$\geq \mu_\theta(y-a) \wedge \mu_\theta(z-b) = \mu(a,y) \wedge \mu(b,z) = \mu_a(y) \wedge \mu_b(z)$$

thus , $\mu_{a+b}(x) \geq \sup_{x=y+z}[\min\{\mu_a(y), \mu_b(z)\}]$,so we have $\mu_{a+b} \geq \mu_a \oplus \mu_b$.

Conversely, for all $x \in R$, if x can be expressible as $x = y+z, (y,z \in R)$,then

$$\mu_a \oplus \mu_b(x) = \max_{x=y+z}[\min\{\mu(a,y), \mu(b,z)\}]$$

$$\geq \max_{\substack{x=y+z \\ \mu(a,y)\neq\mu(b,z)}} [\min\{\mu(a,y), \mu(b,z)\}]$$

$$= \max_{\substack{x=y+z \\ \mu(a,y)\neq\mu(b,z)}} [\min\{\mu_\theta(y-a), \mu(z-b)\}]$$

$$= \max_{\substack{x=y+z \\ \mu(a,y)\neq\mu(b,z)}} [\mu_\theta(y+z)-(a+b)]$$

$$= \mu(a+b, y+z) = \mu(a+b, x).$$

Then, $\mu_a \oplus \mu_b = \mu_{a+b}$.

Similarly, we have the following

Proposition 2.7 Let μ be a fuzzy congruence relation on R. Then, $\mu_a \circ \mu_b \subseteq \mu_{ab}$.

Therefore , we define the binary operation $*$ on R/μ as follows:

$$\mu_a * \mu_b = \mu_{ab}$$

By Proposition2.6 and Proposition2.7 we have the following:

Theorem2.8 Let μ be a fuzzy congruence relation on a ring R. Then $(R/\mu,\oplus,*)$ is a ring.

Proposition2.9: Let μ be a fuzzy congruence relation on R .Then

$\mu^{-1}(1)=\{(a,b)|\mu(a,b)=1,a,b\in R\}$ is a congruence relation on R.

Proof: By Lemma 2.4 in [1], $\mu^{-1}(1)$ is a fuzzy equivalence relation on R and

$$(ax,bx)\in\mu^{-1}(1),\ (xa,xb)\in\mu^{-1}(1).$$

Moreover, for $x\in R$, $\mu(a+x,b+x)\geq\mu(a,b)=1$, which implies that

$\mu(a+x,b+x)=1$,that is $(a+x,b+x)\in\mu^{-1}(1)$. Thus $\mu^{-1}(1)$ is a congruence relation on R .

3. Homomorphism Theorems

Let R and $\overline{R}$ be two rings and f a homomorphism of R to $\overline{R}$. Then, the relation.

$$\ker(f)=\{(a,b)|f(a)=f(b),a,b\in R\}$$

is a congruence relation on R . Then, the characteristic function $\chi_{\ker(f)}$ is a fuzzy congruence relation on R. and

$$\chi_{\ker(f)}(a,b)=\begin{cases}1 & if\ \ f(a)=f(b)\\ 0 & if\ \ f(a)\neq f(b)\end{cases}$$

Theorem 3.1 Let μ be a fuzzy congruence relation on R . and

Let $(R/\mu,\oplus,*)$ be a ring .The mapping $\mu^{\#}:R\rightarrow R/\mu$ defined by,

for $a\in R$, $\mu^{\#}(a)=\mu_a$.

Then $\mu^{\#}$ is a homomorphism.

Proof: It is dear.

Theorom 3.2 Let R and $\overline{R}$ be two rings and $f:R\ \rightarrow\ \overline{R}$ a homomorphism,

Then the fuzzy kernel $\chi_{\ker(f)}$ is a fuzzy congruence on R, and there is a homomorphism

$g:R/\chi_{\ker(f)}\rightarrow\overline{R}$ such that $f=g\,o(\chi_{\ker(f)})^{\#}$.

Proof: Let $a,b\in R$, By the definition $\mu^{\#}$, we have

$$\mu^{\#}(a+b)=\mu_{a+b}=\mu_a\oplus\mu_b=\mu^{\#}(a)\oplus\mu^{\#}(b)$$

and $\mu^{\#}(ab)=\mu_{ab}=\mu_a*\mu_b=\mu^{\#}(a)*\mu^{\#}(b)$.

Now we define $g : R/\chi_{\ker(f)} \to \overline{R}$ by $g((\chi_{\ker(f)})_a) = f(a)$ for all $a \in R$.

If for all $a,b \in R$, $(\chi_{\ker(f)})_a = (\chi_{\ker(f)})_b$, then $\chi_{\ker(f)}\ (a,b) = 1$.

So $(a,b) \in \ker(f)$. Thus we have

$$g((x_{\ker(f)})_a) = f(a) = f(b) = g((x_{\ker(f)})_b),$$

This means that g is well-defined.

If $f(a) = f(b)$, then $(a,b) \in \ker(f)$, $x_{\ker(f)}\ (a,b) = 1$. It implies that

$(x_{\ker(f)})_a = (x_{\ker(f)})_b$, g is one-to-one. Let $a,b \in R$,

$g((x_{\ker(f)})_a \oplus (x_{\ker(f)})_b) = g((x_{\ker(f)})_{a+b})$

$= f(a+b) = f(a) + f(b)$

$= g((x_{\ker(f)})_a) + g((x_{\ker(f)})_b)$

$= g((x_{\ker(f)})_a * (x_{\ker(f)})_b) = g((x_{\ker(f)})_{ab})$

$= f(ab) = f(a)f(b) = g((x_{\ker(f)})_a) \bullet g((x_{\ker(f)})_b)$

Then g is a homomorphism.

Let $a \in R$, $g((x_{\ker(f)})^{\#})(a) = g((x_{\ker(f)})_a) = f(a)$.

So we obtain that

$$g \circ (x_{\ker(f)})^{\#} = f.$$

Theorem 3.3 Let μ and ν be fuzzy congruence relations on R . such that $\mu \subseteq \nu$. Then there is on unique homomorphism

$$g : R/\mu \to R/\nu \qquad \text{such that} \quad g \circ \mu^{\#} = \nu^{\#}.$$

and $\dfrac{R/\mu}{\chi_{\ker(g)}}$ is isomorphic to R/ν.

Proof: Define $g : R/\mu \to R/\nu$ by setting

$$g(\mu_a) = \nu_a \qquad \text{for all } a \in R.$$

Assume that $\mu_a = \mu_b$, then, $1 = \mu(a,b) \leq \nu(a,b)$.

So $\nu(a,b) = 1$, that is, $\nu_a = \nu_b$, then g is well-defined .

The remainder of the proof is clear.

References

[1] N. Kuroki, Fuzzy congruences and fuzzy normal subgroups.Inform.Sci.1992,60:247-259.

[2] N.Kuroki, Fuzzy congruences on inverse semigroups. Fuzzy Sets and Systems,1997, 87:335-340.

[3] N.Kuroki,Fuzzy ideals and bi-ideals in semigroups, Fuzzy Sets and Systems,1981,5:203-215.

[4]F.AL-Thukair,Fuzzy congruence pairs of inverse semigroups, Fuzzy Sets and Systems,1993, 56:117-122.

[5]Zhang Chengyi etc,On the Equal-height Elements of the Fuzzy Subgroups,Chinese Quarterly Journal of Mathmatics,2001,16(2):82-85.