The decision theorems that the equation type Ⅲ of a fuzzy matrix has a solution when the index is one

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Abstract . In this paper, the decision methods that the equation type Ⅲ of a fuzzy matrix has a solution when the index is one were studed, and we had obtained a series of decision theorems that the equation type Ⅲ of a fuzzy matrix has a solution when the index is one.

Keywords. equation type Ⅲ of a fuzzy matrix, decision theorem that the equation type Ⅲ of a fuzzy matrix has the solutions when the index is one.

In this paper, we consider the necessry and sufficient conditions, which the equation type Ⅲ of a fuzzy matrix has a solution when the index is one.

Theorem 1. (**The first decision theorem that the equation type Ⅲ of a fuzzy matrix has a solution when the index is one**) The equation type Ⅲ of a fuzzy matrix $B=(b_{ij})_{n\times n}$ has a solution when the index is one if and only if the unique solution matrix of this equation is that

$$A=(b_{1n}\quad b_{2n-1}\quad \cdots\quad b_{n-12}\quad b_{n1})^{T}.$$

Proof. Necessity. Because the equation type Ⅲ of B has a solution when the index is one, may let $B=(x_1\cdots x_n)^T(x_n\cdots x_1)$, and so

$$\begin{cases} x_1\ x_n = b_{11} \\ x_1\ x_{n-1} = b_{12} \\ \cdots\cdots\cdots \\ x_1\ x_2 = b_{1n-1} \\ x_1 = b_{1n} \\ x_2\ x_n = b_{21} \\ x_2\ x_{n-1} = b_{22} \\ \cdots\cdots\cdots \\ x_2 = b_{2n-1} \\ \cdots\cdots\cdots \\ x_{n-1}\ x_n = b_{n-11} \\ x_{n-1} = b_{n-12} \\ x_n = b_{n1} \end{cases}$$

Therefore we obtaine that

$$(x_1\ x_2\cdots\ x_{n-1}\ x_n)^T=(b_{1n}\ b_{2n-1}\cdots\ b_{n-1\,2}\ b_{n1})^T.$$

And this solution is the unique solution of the equation type Ⅲ of B when the index is one.

Sufficiency. Obviously this conclusion is right.

Theorem 2. Let $B=(b_{ij})\in L^{n\times n}$ is a non-zero sub-symmtric fuzzy square matrix. The equation type Ⅲ of B^* has the solutions when the index is one, if and only if B has the unique linearly independent row vector, and B has the unique linearly independent column vector.

Proof. Necessity. Because the equation type Ⅲ of B has a solution when the index is one, may let that

$$\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{11} \end{bmatrix} = \begin{bmatrix} a_1 \\ \cdots \\ a_n \end{bmatrix} (a_n \cdots a_1) = \begin{bmatrix} a_1(a_n \cdots a_1) \\ \cdots\cdots \\ a_n(a_n \cdots a_1) \end{bmatrix}.$$

If $a_k = \max\{a_1, \cdots, a_n\}$, then the row vectors of B are that

$$\begin{array}{lll} \text{the } 1-\text{st} & \text{row} & a_1(a_n \cdots a_1) = a_1 a_k(a_n \cdots a_1) \\ \cdots\cdots & & \cdots\cdots \\ \text{the } k-\text{th} & \text{row} & a_k(a_n \cdots a_1) \\ \cdots\cdots & & \cdots\cdots \\ \text{the } n-\text{th} & \text{row} & a_n(a_n \cdots a_1) = a_n a_k(a_n \cdots a_1) \end{array}$$

Since B is a non-zero matrix, then $a_k(a_n \cdots a_1)$ is a non-zero vector, too. Therefore $a_k(a_n \cdots a_1)$ is the unique linearly independent row vector of B.

To prove analogously that B has the unique linearly independent column vector.

Sufficiency. Suppose that $(b_{i1} \cdots b_{1,n-i+1})$ $(1 \leqslant i \leqslant n)$ is the unique linearly independent row vector of B, then there is $k_1, \cdots, k_n \in L$, such that

$$(b_{h1} \cdots b_{1,n-h+1}) = k_h(b_{i1} \cdots b_{1,n-i+1}), h = 1, 2, \cdots, n$$

and so

$$B = \begin{bmatrix} k_1(b_{i1} \cdots b_{1,n-i+1}) \\ \cdots\cdots \\ k_n(b_{i1} \cdots b_{1,n-i+1}) \end{bmatrix} = \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} (b_{i1} \cdots b_{1,n-i+1}) = \left[\begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} b_{i1}, \cdots, \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} b_{1,n-i+1} \right] \quad (1)$$

If $b = \max\{b_{i1}, \cdots, b_{i,n-i+1}, \cdots, b_{1,n-i+1}\}$, then the column vectors of B are that

$$(2) \quad \begin{cases} \text{the } 1-\text{st} & \text{column} & b_{i1}(k_1 \cdots k_n)^T = b_{i1} b(k_1 \cdots k_n)^T \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \text{the } (n-i+1)-\text{th} & \text{column} & b_{i,n-i+1}(k_1 \cdots k_n)^T = b_{i,n-i+1} b(k_1 \cdots k_n)^T \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \text{the } n-\text{th} & \text{column} & b_{1,n-i+1}(k_1 \cdots k_n)^T = b_{1,n-i+1} b(k_1 \cdots k_n)^T \end{cases}$$

Now B is a non-zero matrix, then $b(k_1 \cdots k_n)^T$ is a non-zero vector, too. Therefore $b(k_1 \cdots k_n)^T$ already is the unique column vector of the linear independence of B.

Because B is a fuzzy sub-symmtric square matrix, if the i-th row vector of B is the unique row vector of the linear independence of B, then the $(n-i+1)$-th column vector of B is the unique column vector of the linear independence of B.

Therefore

$$(b_{1,n-i+1} \cdots b_{i,n-i+1} \cdots b_{i1})^T \neq b(k_1 \cdots k_n)^T = (bk_1 \cdots bk_n)^T \quad (3)$$

By the expressions(1)～(3), we have that

$$B = \left[\begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} b_{i1}, \cdots, \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} b_{1,n-i+1} \right] = \left[\begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} b\, b_{i1}, \cdots, \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} b\, b_{1,n-i+1} \right] = \left[\begin{bmatrix} bk_1 \\ \cdots \\ bk_n \end{bmatrix} b_{i1}, \cdots, \begin{bmatrix} bk_1 \\ \cdots \\ bk_n \end{bmatrix} b_{1,n-i+1} \right]$$

$$= \left[\begin{bmatrix} b_{1,n-i+1} \\ \cdots \\ b_{i1} \end{bmatrix} b_{i1}, \cdots, \begin{bmatrix} b_{1,n-i+1} \\ \cdots \\ b_{i1} \end{bmatrix} b_{1,n-i+1} \right] = \begin{bmatrix} b_{1,n-i+1} \\ \cdots \\ b_{i1} \end{bmatrix} (b_{i1} \cdots b_{1,n-i+1}) = A\ A^{ST},$$

where $A = (b_{1,n-i+1}, \cdots, b_{i1})^T$. Therefore the equation type Ⅲ of B has a solution when the index is one. Therefore we again have obtanined that

Theorem 3. let B is a non − zero fuzzy sub − symmtric square matrix. The equation type Ⅲ of B has a solution when the index is one if and only if B is equal to product of both of the unique linearly independent column vector of B and the unique linearly independent row vector of B.

It must be pointed out that by Definition 2 in〔2〕, the unique linearly independent column vector of B exactly is the maximum column vector of B, and the unique linearly independent row vector of B exactly is the maximum row vector of B, too.

Theorem 4. (The second decision theorem that the equation type Ⅲ of a fuzzy matrix has a solution when the index is one) Let B is a non − zero fuzzy sub − symmtric square matrix. The equation type Ⅲ of B has a solution when the index is one if and only if B is equal to a product of both of the maximum column vector of B and the maximum row vector of B.

By Theorem 1 and Theorem 4 we have direct gained that

Theorem 5. let B is a non − zero fuzzy sub − symmtric square matrix. If the equation type Ⅲ of B has a solution when the index is one, then the maximum column vector of B is $(b_{1n}\ b_{2n-1} \cdots b_{n-1\,2}\ b_{n1})^T$, and the maximum row vector of B is $(b_{n1}\ b_{n-1\,2} \cdots b_{2n-1}\ b_{1n})$, and the maximum column vector of B is the unque solution matrix that the equation type Ⅲ of B has a solution when the index is one.

Definition 1.〔3〕 If a row (or column) of a fuzzy matrix A has identical elements, then this row (column) of A is called the same element row (or column) of A. If the minimal element of A forms a same element row (or column) of A, the we say that the row (or column) of A can be canceled, and other rows (or columns) of A forms a new matrix, which is called a submatrix of A. And this cancellation goes on step by step till it can not be canceled any more, which is called gradual cancellation, till the last submatrix is come out.

Theorem 6 (The third decision theorem that the equation type Ⅲ of a fuzzy matrix has a solution when the index is one) Let B is a non − zero fuzzy sub − symmetric square matrix. The equation type Ⅲ of B has a solution when the index is one if and only if the rows (or columns) to B and theirs submatrices can be canceled step by step till comes a 1×1 submatrix or a $u \times v (u \leqslant n, v \leqslant n)$ submatrix (theirs elements are maximal element of B). And a solution matrix is the column vector containing maximal element of B.

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