Fuzzy Pre-Urysohn Spaces *

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Abstract: *In this paper, the new concept of fuzzy pre-Urysohn spece in fuzzy topological spaces is introduced. It is a kind of new fuzzy space bettwen fuzzy weakly Urysohn spece and fuzzy PS-Urysohn Space. Its characteristic properties and basic properties are studied.*

Keywords: *Fuzzy topological space; Remote-neighborhood; Preclosed set; Pre-Urysohn Space*

1 Introduction

Urysohn space[5] is one of the important notions in topology. Chen and Xiao introduced the fuzzy Urysohn space[7] and fuzzy weakly Urysohn space[8] in fuzzy topological spaces, respectively. And we introduced the fuzzy PS-Urysohn space in fuzzy topological spaces in [4]. The main purpose of this paper is to introduce and study the concept of the fuzzy pre-Urysohn space in fuzzy topological spaces with the help of the remote-neighborhood[11] and fuzzy preinterior of fuzzy sets[3]. The fuzzy pre-Urysohn space is a kind of new fuzzy space bettwen fuzzy weakly Urysohn spece and fuzzy PS-Urysohn Space. And some of its good results are obtained.

2 Preliminaries

By a fuzzy topological space(fts, for short), we mean the pair (X, δ) where X is a nonempty crisp set and δ a subset of I^X satisfying the open set axiom as usually mentioned in general topology, i.e., we follow the definition of fuzzy topological space given in [6]. For a fuzzy set A in X, the notations A^o, A^-, A_o, A_- and A'

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will respectively stand for the interior, closure, semiinterior, semiclosureand and complement of A.

Definitoin 2.1 (Bai [1,2]). Let (X, δ) be an fts and A be a fuzzy set in X. Then A is said to be

(1) Fuzzy strongly semiopen set iff there is a $B \in \delta$ such that $B \leq A \leq B^{-o}$.

(2) Fuzzy strongly semiclosed set iff there is a $B' \in \delta$ such that $B^{o-} \leq A \leq B$.

(3) Fuzzy pre-semiopen set iff $A \leq (A^{-})_{o}$.

(4) Fuzzy pre-semiclosed set iff $A \geq (A^{o})_{-}$.

Definition 2.2(Bai [1,2]). Let (X, δ) be an fts and A be a fuzzy set in X. Then

$A^{\Delta} = \bigcup\{B : B \leq A, B \text{ fuzzy strongly semiopen }\}$,

$A^{\sim} = \bigcap\{B : A \leq B, B \text{ fuzzy strongly semiclosed }\}$

$A_{\Delta} = \bigcup\{B : B \leq A, B \text{ fuzzy pre-semiopen }\}$,

$A_{\sim} = \bigcap\{B : A \leq B, B \text{ fuzzy pre-semiclosed }\}$

are called the strong semiinterior, strong semiclosure, pre-semiinterior and pre-semiclosure of A, respectively.

Definitoin 2.3 (Singal [10]). Let (X, δ) be an fts and A be a fuzzy set in X. Then A is said to be

(1) Fuzzy preopen set iff $A \leq A^{-o}$.

(2) Fuzzy preclosed set iff $A \geq A^{o-}$.

$PO(X)$ and $PC(X)$ will always denote the family of fuzzy preopen sets and family of fuzzy preclosed sets of an fts (X, δ), respectively. Obviously, $A \in PO(X)$ iff $A' \in PC(X)$.

Definition 2.4(Bai [3]). Let (X, δ) be an fts and A be a fuzzy set in X. Then

$A_{\square} = \bigcup\{B : B \in PO(X), B \leq A\}$,

$A_{\frown} = \bigcap\{B : B \in PC(X), A \leq B\}$

are called the preinterior and preclosure of A, respectively.

Remark 2.5. Every fuzzy open set is a fuzzy strongly semiopen set, every fuzzy strongly semiopen set is a fuzzy preopen set and every fuzzy preopen set is a fuzzy pre-semiopen set[1,2]. That none of the converses need be true[1,2].

Obviously,

$A^{o} \leq A^{\Delta} \leq A_{\square} \leq A_{\Delta} \leq A$.

$A \leq A_{\sim} \leq A_{\frown} \leq A^{\sim} \leq A^{-}$.
$A \leq B \Rightarrow A_{\Box} \leq B_{\Box}; A_{\frown} \leq B_{\frown}$.
$0_{\Box} = 0; 0_{\frown} = 0; 1_{\Box} = 1; 1_{\frown} = 1$.
$(A_{\Box})_{\Box} = A_{\Box}; (A_{\frown})_{\frown} = A_{\frown}$.
$A \in PO(X)$ iff $A_{\Box} = A$.
$A \in PC(X)$ iff $A_{\frown} = A$.

Definition 2.6(Pu [9]). Let (X, δ) be an fts. A fuzzy net $S = \{S(n), n \in D\}$ in (X, δ) is a function $S : D \rightarrow \xi$ where D is a directed set with order relation $\geq$ and ξ the collection of all the fuzzy points in X.

Definition 2.7 (Wang [11]). Let (X, δ) be an fts, x_λ be a fuzzy point and P a fuzzy closed set in X. Then P is called a remoted-neighborhood of x_λ, if $x_\lambda \notin P$. The set of all remote-neighborhoods of x_λ will be denoted by $\eta(x_\lambda)$.

Definitoin 2.8. An fts (X, δ) is called a fuzzy Hausdorff space (fuzzy Urysohn space, fuzzy weakly Urysohn space, fuzzy PS-Urysohn space) If for every pair fuzzy points x_λ and y_μ with $x \neq y$, there exist $P \in \eta(x_\lambda)$ and $Q \in \eta(y_\mu)$ such that $P \cup Q = 1$ ($P^o \cup Q^o = 1$, $P^\Delta \cup Q^\Delta = 1$, $P_\Delta \cup Q_\Delta = 1$)[4,7,8,12].

3 Fuzzy Pre-Urysohn Spaces

Definitoin 3.1. Let (X, δ) be an fts. If for every pair fuzzy points x_λ and y_μ with $x \neq y$, there exist $P \in \eta(x_\lambda)$ and $Q \in \eta(y_\mu)$ such that $P_{\Box} \cup Q_{\Box} = 1$, then (X, δ) is said to be fuzzy pre-Urysohn space.

Corollary 3.2. Fuzzy Urysohn space
$\Rightarrow$ fuzzy weakly Urysohn space
$\Rightarrow$ fuzzy pre-Urysohn space
$\Rightarrow$ fuzzy PS-Urysohn space
$\Rightarrow$ fuzzy Hausdorff space.

Proof. This is immediate from Definitions 3.1, 2.8.

Definition 3.3. An fts (X, δ) is called preinterior additive if $(A \cup B)_{\Box} = A_{\Box} \cup B_{\Box}$ for any two fuzzy sets A and B in X.

Theorem 3.4. Let (X, δ) be a fuzzy pre-Urysohn space. Then every preinterior additive subspace $(Y, \delta \mid Y)$ in (X, δ) is also a fuzzy pre-Urysohn space.

Proof. Let x_λ and y_μ be two fuzzy points in $(Y, \delta \mid Y)$ with $x \neq y$, $x^*{}_\lambda$ and $y^*{}_\mu$ be extensions of x_λ and y_μ in (X, δ), respectively. Since (X, δ) is a fuzzy pre-Urysohn space and $x \neq y$, there exist $P \in \eta(x^*{}_\lambda)$ and $Q \in \eta(y^*{}_\mu)$ such that $P_\Box \cup Q_\Box = 1$. Again, by $(Y, \delta \mid Y)$ is preinterior additive and Theorem —2.7.3 in [12], we have

$$(P \mid Y)_\Box \cup (Q \mid Y)_\Box = ((P \mid Y) \cup (Q \mid Y))_\Box$$
$$= ((P \cup Q) \mid Y)_\Box \geq ((P_\Box \cup Q_\Box) \mid Y)_\Box$$
$$= (1 \mid Y)_\Box = 1_Y.$$

Thus, $(Y, \delta \mid Y)$ is a fuzzy pre-Urysohn space.

Lemma 3.5. Let $f : (X, \delta) \to (Y, \tau)$ be a fuzzy homeomorphic mapping[12] from a fuzzy space (X, δ) to another fuzzy space (Y, τ). If A is a fuzzy preopen set of (X, δ), then $f(A)$ is a fuzzy preopen set of (Y, τ).

Proof. Let A is a fuzzy preopen set of (X, δ). Then $A \leq A^{-o}$. Hence, $f(A) \leq f(A^{-o})$. Since f is a fuzzy homeomorphic mapping,

$$f(A^{-o}) = (f(A))^{-o}, \text{ i.e., } f(A) \leq (f(A))^{-o}.$$

Thus, $f(A)$ is a fuzzy preopen set of (Y, τ).

Theorem 3.6. Let $f : (X, \delta) \to (Y, \tau)$ be a fuzzy homeomorphic mapping from a fuzzy pre-Urysohn space (X, δ) to another fuzzy space (Y, τ). Then (Y, τ) is also a fuzzy pre-Urysohn space.

Proof. Let y_λ and $y^*{}_\mu$ be two fuzzy points in (Y, τ) with $y \neq y^*$. Then there are two fuzzy points x_λ and $x^*{}_\mu$ in (X, δ) with $x \neq x^*$ such that $f(x_\lambda) = y_\lambda$ and $f(x^*{}_\lambda) = y^*{}_\lambda$. Since (X, δ) is a fuzzy pre-Urysohn space, there exist $P \in \eta(x_\lambda)$ and $Q \in \eta(x^*{}_\lambda)$ such that $P_\Box \cup Q_\Box = 1$. Again, since f is a fuzzy homeomorphic mapping, we have $f(P) \in \eta(y_\lambda)$ and $f(Q) \in \eta(y^*{}_\mu)$. By Lemma 3.5

$$(f(P))_\Box \cup (f(Q))_\Box \geq (f(P_\Box))_\Box \cup (f(Q_\Box))_\Box$$
$$= f(P_\Box) \cup f(Q_\Box) = f(P_\Box \cup Q_\Box)$$
$$= f(1) = 1.$$

Thus, (Y, τ) is a fuzzy pre-Urysohn space.

4 The Characterizations of Fuzzy Pre-Urysohn Spaces

Theorem 4.1. A fuzzy topological space (X, δ) is a fuzzy pre-Urysohn space iff for every pair fuzzy points x_λ and y_μ with $x \neq y$ and $\lambda < 1, \mu < 1$, there exist $U \in \delta$ and $V \in \delta$ such that $x_\lambda \in U, y_\mu \in V$ and $U_\frown \cap V_\frown = 0$.

Proof. Let (X, δ) be a fuzzy pre-Urysohn space, x_λ and y_μ be two fuzzy points in X with $x \neq y$ and $\lambda < 1, \mu < 1$. Choose two real numbers r and s satisfying $0 < r < 1 - \lambda$ and $0 < s < 1 - \mu$. Then there exist $P \in \eta(x_r)$ and $Q \in \eta(y_s)$ such that $P_\square \cup Q_\square = 1$. Put $U = P'$ and $V = Q'$. Then $U \in \delta$ and $V \in \delta$ and $x_\lambda \in U, y_\mu \in V$ and

$U_\frown \cap V_\frown = (P')_\frown \cap (Q')_\frown = (P_\square)' \cap (Q_\square)' = (P_\square \cup Q_\square)' = 0.$

Conversely, let the given condition hold. Suppose x_λ and y_μ are two fuzzy points with $x \neq y$. Choose two real numbers r and s satisfying $1 - \lambda < r < 1$ and $1 - \mu < s < 1$. By hypothesis there exist $U \in \delta$ and $V \in \delta$ such that $x_r \in U, y_s \in V$ and $U_\frown \cap V_\frown = 0$. Put $P = U'$ and $Q = V'$. Then $P \in \eta(x_\lambda), Q \in \eta(y_\mu)$ and

$P_\square \cup Q_\square = (U')_\square \cup (V')_\square = (U_\frown)' \cup (V_\frown)' = (U_\frown \cap V_\frown)' 1.$

Thus, (X, δ) is a fuzzy pre-Urysohn space.

Definitoin 4.2. Let (X, δ) be an fts, $S = \{S(n), n \in D\}$ a fuzzy net in X and x_λ a fuzzy point. Then x_λ is said to be a PW-limit point of S (or S PW-converges to x_λ), if for each $P \in \eta(x_\lambda), S(n) \not\leq P_\square$ is eventually true.

Theorem 4.3. An fts (X, δ) is a fuzzy pre-Urysohn space iff no fuzzy net in X can PW-converge to two fuzzy points x_λ and y_μ with $x \neq y$.

Proof. Let (X, δ) be a fuzzy pre-Urysohn space, $S = \{S(n), n \in D\}$ be a fuzzy net in X which PW-converges to a fuzzy point x_λ, and y_μ be another with $x \neq y$. Then there exist $P \in \eta(x_\lambda)$ and $Q \in \eta(y_\mu)$ such that $P_\square \cup Q_\square = 1$. Since $S(n) \not\leq P_\square$ is eventually true, therefore, $S(n)$ is eventually in $Q_\square$. Thus, S does not PW-converge to y_μ.

Conversely, assume that the condition is true and that x_λ and y_μ are two fuzzy points with $x \neq y$. If for every $P \in \eta(x_\lambda)$ and $Q \in \eta(y_\mu)$, $P_\square \cup Q_\square \neq 1$, then there exists a fuzzy point $S(P, Q) \notin P_\square \cup Q_\square$. Take

$S = \{S(P, Q) : (P, Q) \in \eta(x_\lambda) \times \eta(y_\mu)\}.$

Then S is a net in X With the following relation:

$(P_1, Q_1) \leq (P_2, Q_2)$ iff $P_1 \leq P_2$ and $Q_1 \leq Q_2$
Where $(P_1, Q_1), (P_2, Q_2) \in \eta(x_\lambda) \times \eta(y_\mu)$. Obviously, eventually $S(n) \not\leq P_\Box$, so S PW-converges to x_λ. Similarly, S PW-converges to y_μ as well. This contradicts the hypothesis. Consequently,there are $P \in \eta(x_\lambda)$ and $Q \in \eta(y_\mu)$ such that $P_\Box \cup Q_\Box = 1$. Thus, (X, δ) is a fuzzy pre-Urysohn space.

Definition 4.4. An fts (X, δ) is called interior additive if $(A \cup B)^o = A^o \cup B^o$ for any two fuzzy sets A and B in X.

Proposition 4.5. Let (X, δ) be a interior additive fts. Then (X, δ) is a fuzzy Urysohn space iff it is a fuzzy Hausdorff space.

Proof. Necessity. This is immediate from Corollary 3.2.

Sufficiency. Let x_λ and y_μ be two fuzzy points in (X, δ) with $x \neq y$. Since (X, δ) is a fuzzy Hausdorff space, there exist $P \in \eta(x_\lambda)$ and $Q \in \eta(y_\mu)$ such that $P \cup Q = 1$. Again, since (X, δ) is interior additive,

$$P^o \cup Q^o = (P \cup Q)^o = 1.$$

Thus, (X, δ) is a fuzzy Urysohn space.

From Corollary 3.2 and Proposition 4.5 we have following results.

Theorem 4.6. Let (X, δ) be a interior additive fts. Then

(X, δ) is a fuzzy Urysohn space
$\Leftrightarrow$ (X, δ) is a fuzzy weakly Urysohn space
$\Leftrightarrow$ (X, δ) is a fuzzy pre-Urysohn space
$\Leftrightarrow$ (X, δ) is a fuzzy PS-Urysohn space
$\Leftrightarrow$ (X, δ) is a fuzzy Hausdorff space.

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