The necessary and sufficient condition that a fuzzy set is a pointwise fuzzy group

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Abstract: 1971 Rosenfeld gave the concept of fuzzy group in [1] and 1981 Qi Zhen-Kai Gave the definition of pointwise fuzzy group in [2]. In this paper we gave a necessary and sufficient condition that a fuzzy set is a pointwise fuzzy group.

Keywords: fuzzy set; fuzzy group; pointwise fuzzy group.

1. Preliminaries

Let X be a nonempty set. A fuzzy set A is a map A: $X \rightarrow [0,1]$, and F(X) will denote the fuzzy power set of X.. We first recall some basic definitions and results that will be needed in the sequel.

Definition1.1 [1] Let G be a group and $A \in F(G)$. If
(i) $A(xy) \geqslant \min(A(x), A(y))$, $x, y \in G$;
(ii) $A(x^{-1}) \geqslant A(x)$, $x \in G$,
then A is called a fuzzy group on G.

Definition1.2 Let X be a nonempty set , $x\in X$, $\lambda\in[0,1]$, $x_\lambda\in F(X)$ and $y\in X$, define

$$x_\lambda(y)=\begin{cases}\lambda, & y=x\\ 0, & y\neq x\end{cases}$$

x_λ is called a fuzzy point on X.

Clearly, if $\lambda\neq 0$, $x_\lambda=y_\mu$ iff $x=y$, $\lambda=\mu$; $x_\lambda\subseteq y_\mu$ iff $x=y$, $\lambda\leqslant\mu$.

we define $a_\lambda\in A\Leftrightarrow a_\lambda\subseteq A$

Proposition1.3 $a_\lambda\in A\Leftrightarrow A(a)\geq\lambda$

Proof. If $a_\lambda\in A$, then $\lambda=a_\lambda(a)\leqslant A(a)$. If $A(a)\geqslant\lambda$, then $a_\lambda\subseteq A$ hence $a_\lambda\in A$.

Definition1.4 Let X be a nonempty set and $A\in F(X)$. If there exists a rule, for any a_λ, $b_\mu\in A$, there is only one $c_v\in A$ corresponding to them, $v=\min\{\lambda,\mu\}$, and for any a_s, $b_t\in A$, the element corresponding to them must be $c_r\in A$, $r=\min\{s,t\}$. We call this rule the multiplication of A, c_v the product of a_λ and b_μ, which is written as $c_v=a_\lambda b_\mu$. We also call A forms a pointwise fuzzy groupoid with regard to this multiplication .

Definition1.5 Let A be a pointwise fuzzy groupoid on X. Then A is called a pointwise fuzzy semigroup if, for any a_λ, b_μ, $c_v\in A$,

$$(a_\lambda b_\mu)c_v=a_\lambda(b_\mu c_v).$$

Definition1.6 Let A be a pointwise fuzzy groupoid on X. If there exists $e_\lambda \in A$ satisfying

$$e_\lambda a_\mu = a_\mu \quad \text{for any } a_\mu \in A.$$

Then e_λ is called a left identify of A. Similarly we can define the right identify of A. If e_λ is both a left and a right identify of A, then e_λ is called the identify of A.

Proposition 1.7 If the pointwise fuzzy groupoid A on X has a left identify e_λ and a right identify f_μ, then $e_\lambda = f_\mu$.

Proof. Since e_λ is a left identify, hence

$$e_\lambda f_\mu = f_\mu.$$

Since f_μ is a right identify, hence

$$e_\lambda f_\mu = e_\lambda.$$

Therefore, $e_\lambda = f_\mu$.

Corollary. The identify of pointwise fuzzy groupoint A on X is unique if it exist.

Proposition 1.8 If e_λ is the left identify of pointwise fuzzy groupoid A on X, then $A(x) \leqslant \lambda$ holds for all $x \in X$.

Proof. For all $x \in X$, since e_λ is a left identify, hence

$$e_\lambda x_{A(x)} = x_{A(x)}.$$

Therefore, $A(x) = \min\{A(x), \lambda\}$ by Definition1.3 and $\min\{A(x), \lambda\} \leqslant \lambda$. So that

$A(x) \leqslant \lambda$.

Proposition 1.9 If e_λ is a left identify of pointwise fuzzy groupoid A on X, then $\lambda = A(e)$.

Proof. Since $e_\lambda \in A$, hence $A(e) \geqslant \lambda$ by Proposition1.3. Since e_λ is a left identify and $e \in X$ hence $A(e) \leqslant \lambda$ by

Proposition1.8. Therefore, $A(e)=\lambda$.

Definition 1.10 Let A be a pointwise fuzzy groupoid on X and suppose A has the left identify e_λ. If for $a_\mu \in A$, there exists $b_s \in A$ satisfying $b_s a_\mu = e_\mu$, then b_s is called a left inverse of a_μ. Similarly we can define the right inverse of a_μ.

Definition 1.11 Let A be a pointwise fuzzy semigroup on X. If

(1) there exists a left identify $e_\lambda \in A$, and if

(2) for any $a_\mu \in A$, there is at least a left inverse in A.

Then A is called a pointwise fuzzy group on X.

Example 1.12 Let X be a group and suppose A is a fuzzy group on X (see Definition1.1), then A is a poindwise fuzzy group on X.

Proof. For any $x_\lambda, y_\mu \in A$, let

$$x_\lambda y_\mu = (xy)_v, \quad v = \min\{\lambda, \mu\} \qquad (1\text{-}1)$$

Then A forms a pointwise fuzzy groupoid with respect to the rule (1-1). Since X is a group, hence we have, for any x_λ, y_μ, $z_v \in A$

$$(x_\lambda y_\mu) z_v = (xy)_{\min\{\lambda,\mu\}} z_v = ((xy)z)_{\min\{\min\{\lambda,\mu\},v\}}$$

$$= (x(yz))_{\min\{\lambda,\min\{\mu,v\}\}} = x_\lambda (yz)_{\min\{\mu,v\}} = x_\lambda (y_\mu z_v)$$

so A is a pointwise fuzzy simegroup on X.

Since A is a fuzzy group on X, hence when e is the unit of X, for all $x \in X$

$$A(e) = A(xx^{-1}) \geqslant \min\{A(x), A(x^{-1})\} = A(x).$$

Therefore, we also have, for all $x_\mu \in A$

$$e_{A(e)} x_\mu = (ex)_{\min\{A(e),\mu\}} = x_\mu,$$

so $e_{A(e)}$ is the left identify of A .

If $x_\mu \in A$, in case of $A(x^{-1}) \geqslant A(x) \geqslant \mu$, then $(x^{-1})_\mu \in A$, and

$$(x^{-1})_\mu x_\mu = (x^{-1}x)_{\min\{\mu,\mu\}} = e_\mu,$$

Hence x_μ has a left inverse $(x^{-1})_\mu \in$ A and A is a pointwise fuzzy group on X.

2. The necessary and sufficient condition that a fuzzy set is a pointwise fuzzy group

Theorem 2.1 Let X be a nonempty set and $A \in F(X)$. Then A is a pointwise fuzzy group iff A is a pointwise fuzzy groupoid and in agreement with the following conditions:

(1) $(a_\lambda b_\mu)c_\nu = a_\lambda(b_\mu c_\nu)$ for any a_λ, b_μ, $c_\nu \in A$.

(2) There exists a left identify $e_\lambda \in A$ such that

$$e_\lambda x_\mu = x_\mu \text{ for any } x_\mu \in A.$$

(3) For any $x_\mu \in A$, there exists at least a left inverse $y_\nu \in A$ such that

$$y_\nu x_\mu = e_\mu.$$

Proof. If A is a pointwise fuzzy group, then A is a pointwise fuzzy semigroup by Definition1. 11 and so A is a pointwise fuzzy groupoid and satisfying conditions (1), (2)and (3). If A is a pointwise fuzzy groupoid and in agreement with conditions (1), (2)and(3), then A is a pointwise fuzzy semigroup by Definition1. 5 and A is a pointwise fuzzy group by Definition1. 11.

According to Theorem2. 1 we can give a new definition of pointwise fuzzy group equivalent to Definition1. 11.

Definition2. 2 Let X be a nonempty set and $A \in F(X)$. If there exists a rule, for any a_λ, $b_\mu \in A$, there is only one

$c_v \in A$ corresponding to them, $v=\min\{\lambda, \mu\}$, and for any a_s, $b_t \in A$, the element corresponding to them must be $c_r \in A$, $r = \min\{s, t\}$. We call this rule the multiplication of A, c_v the product of a_λ and b_μ, which is written as $c_v = a_\lambda b_\mu$. And the multiplication and in agreement with the following conditions:

(1) $(a_\lambda b_\mu)c_v = a_\lambda (b_\mu c_v)$ for any a_λ, b_μ, $c_v \in A$.

(2) There exists a left identify $e_\lambda \in A$ such that

$$e_\lambda x_\mu = x_\mu \text{ for any } x_\mu \in A.$$

(3) For any $x_\mu \in A$, there exists at least a left inverse $y_v \in A$ such that

$$y_v x_\mu = e_\mu.$$

Then A is called a pointwise fuzzy group on X.

If X is a group and A is a fuzzy group on X. Let, for any a_λ, $b_\mu \in A$

$$a_\lambda b_\mu = (ab)_{\min\{\lambda, \mu\}}$$

Then A is a pointwise fuzzy group on X by Example1. 12.

Theorem2. 3 Let X be a nonempty set and A a pointwise fuzzy semigroup on X. Then A is a pointwise fuzzy group on X iff for any a_λ, $b_\mu \in A$, $v=\min\{\lambda, \mu\}$,

$$a_v x_v = b_v \text{ and } y_v a_v = b_v$$

has a solution in A.

Proof. First we prove the sufficient. For any $a_\lambda \in A$, since

$$y_\lambda a_\lambda = a_\lambda$$

has a solution in A, hence there is a $e_\lambda \in A$ satisfying

$$e_\lambda a_\lambda = a_\lambda$$

For another any $b_\mu \in A$, let $v = \min\{\lambda, \mu\}$. Since

$$a_v x_v = b_v$$

has a solution in A , so there exists a $c_v \in A$ satisfying

$$a_v c_v = b_v.$$

Hence

$$e_v b_v = e_v (a_v c_v) = (e_v a_v) c_v$$
$$= a_v c_v = b_v$$

Assume $A(e)=\xi$. It is easy to get

$$e_\xi a_\lambda = a_\lambda \qquad \text{for all } a_\lambda \in A.$$

Hence $e_\xi \in A$ is a left identify .

Second, for each $a_\lambda \in A$, then $e_\lambda \in A$, and according to the hypothesis

$$y_\lambda a_\lambda = e_\lambda$$

has a solution $b_\lambda \in A$ such that $b_\lambda a_\lambda = e_\lambda$

so a_λ has a left inverse $b_\lambda \in A$. Therefore, A is a pointwise fuzzy group on X.

Now we come to prove the necessary. Since A is a pointwise fuzzy group on X, hence for any a_λ, $b_\mu \in A$, here exist $c_\xi \in A$ such that $c_\xi a_\lambda = e_\lambda$, $e_{A(e)}$ is the identify of A.

Let $b_\mu c_\xi = d_\zeta$. Then $\zeta = \min\{\mu, \xi\} \leqslant \mu$, $\lambda = \min\{\xi, \lambda\} \leqslant \xi$.

If $v = \min\{\lambda, \mu\}$, then

$$d_v a_v = (b_v c_v) a_v = b_v (c_v a_v) = b_v e_v = b_v.$$

Therefore, $y_v a_v = b_v$ has a solution $d_v \in A$. Similarly can prove $a_v x_v = b_v$ has a solution in A.

References

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