The Base of Finite EI Algebra[1]

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Abstract: In this paper, we give the definition of base of the EI algebra and EI independence and apply them to study the algebraic structures of the EI algebra. We also give some theorem to find the base for some EI algebra.

Key words: EI algebra; EI independent; Base of EI algebra

It is well known that fuzziness is one of the important characteristics of human cognizance and thinking. The theory of fuzzy sets and systems have been applied in many fields, especially in fuzzy information processing since it was proposed by professor L. A. Zadeh in [1] 1965. Many mathematicians, engineers and technicians have achieved many important results by using and applying fuzzy theory. Because some methods of determining membership functions lack of mathematical strictness and unification and different persons have different membership function for the same fuzzy concept, many powerful mathematical tools can not exert the effects they should produce in fuzzy information processing. In [2, 3, 4, 5], the author have studied the AFS (Axiomatic Fuzzy Sets) theory based on some new mathematical objects such as AFS algebra which is molecular lattices [11], AFS structure which is a special system (system is one of main mathematical objects in combinatorics [7]) and cognitive fields. In this paper, by the definition of base of the EI algebra and EI independence, we study the algebraic structures of the EI algebra—one kind of AFS algebra.

In the following, we introduce the AFS algebra. Let M be a set.

$$M^*=\{\sum_{i\in I}A_i \mid A_i\in 2^M,\ i\in I,\ I \text{ is any indexing set}\},$$

when I is finite set, $\sum_{i\in I}A_i$ is also denoted as $A_1+A_2+\ldots+A_n$. $\sum_{i\in I}A_i$ is just sum in form, and these A_i in $\sum_{i\in I}A_i$ can be in any order. For example, $\sum^2_{i=1}A_i=A_1+A_2=A_2+A_1$. In [2,4], the author has defined the equivalence relation R in M^* as following: $\alpha=\sum_{i\in I}A_i$, $\beta=\sum_{j\in J}B_j\in M^*$, $\alpha R\beta \Leftrightarrow \forall A\in\{A_i \mid i\in I\}$, $\exists B\in\{B_j \mid j\in J\}$ such that $A\supseteq B$ and $\forall B\in\{B_j \mid j\in J\}$, $\exists A\in\{A_i \mid i\in I\}$ such that $B\supseteq A$. Without confusion, we always denote M^*/R as EM. In the following, α, $\beta\in EM$, $\alpha=\beta$ means that α and β are equivalent. Under the equivalence relation R, if $A_u\supseteq A_v$, u, $v\in I$, then

$$\sum_{i\in I}A_i=\sum_{i\in I,\ i\neq u}A_i \tag{1}$$

In [4], the author has proved that $(EM, \wedge, \vee)$ is molecular lattices if the lattice operators $\wedge$ and $\vee$ are defined as following: $\sum_{i\in I}A_i$, $\sum_{j\in J}B_j\in EM$,

$$\sum_{i\in I}A_i\vee\sum_{j\in J}B_j=\sum_{k\in U}C_k,\quad \sum_{i\in I}A_i\wedge\sum_{j\in J}B_j=\sum_{i\in I,\, j\in J}A_i\cup B_j, \tag{2}$$

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where $U=I\coprod J$ is the disjoin union of set I and J, $k\in U$, $C_k=A_k$ when $k\in I$; $C_k=B_k$, when $k\in J$. $(EM, \wedge, \vee)$ is called the EI algebra over M. EI is a kind of AFS algebra.

Definition 1 Let M be a set, EM be the EI algebra over M. $S\subseteq EM$, $(S, \wedge, \vee)$ is called a sub-algebra of $(EM, \wedge, \vee)$ if for any α, $\beta\in S$, (1) $\alpha\vee\beta\in S$, (2) $\alpha\wedge\beta\in S$.

Proposition 1 Let M be a set, EM be the EI algebra over set M, $\Lambda\subseteq EM$. If

$$(\Lambda)_{EI}=\{\vee_{i\in I}(\wedge_{\gamma\in T_i}\gamma)\mid T_i\subseteq\Lambda, i\in I, I \text{ is any indexing set}\},$$

then $(\Lambda)_{EI}$ is the sub-algebra of EM. $(\Lambda)_{EI}$ is called sub-algebra of EM generated by Λ.

Proof Since $\Lambda\subseteq EM$, hence $(\Lambda)_{EI}\subseteq EM$. For any $\alpha,\beta\in(\Lambda)_{EI}$, $\alpha=\vee_{i\in I}(\wedge_{\gamma\in T_i}\gamma)$,

$\beta=\vee_{j\in J}(\wedge_{\gamma\in H_j}\gamma)$, since $T_i\cup H_j\subseteq\Lambda$, hence $\alpha\wedge\beta=\vee_{i\in I,j\in J}(\wedge_{\gamma\in T_i\cup H_j}\gamma)\in(\Lambda)_{EI}$.

$\alpha\vee\beta=(\vee_{i\in I}(\wedge_{\gamma\in T_i}\gamma))\vee(\vee_{j\in J}(\wedge_{\gamma\in H_j}\gamma))\in(\Lambda)_{EI}$. Therefore $(\Lambda)_{EI}$ is the sub-algebra of EM.

Definition 2 Let M be a finite set, EM be the EI algebra of M, $D=\{\alpha_1, \alpha_2, \dots, \alpha_n\}\subseteq EM$. $\alpha_1, \alpha_2, \dots, \alpha$ are called EI independent if $\forall\alpha_i\in D$, $\alpha_i\notin(D\backslash\{\alpha_i\})_{EI}$, otherwise $\alpha_1, \alpha_2, \dots, \alpha_n$ are called EI dependent.

Definition 3 Let M be a finite set, $S\subseteq EM$, S is a sub-algebra of EM, $\Lambda\subseteq S$, $\Lambda=\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, $\Lambda=\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is called a base of EI finite Algebra of S if 1) $(\Lambda)_{EI}=S$, 2) $\alpha_1, \alpha_2, \dots, \alpha_n$ are EI independent.

Definition 4 Let M be a finite set. $\alpha=\Sigma_{i\in I}A_i$, $\alpha\in EM$, and α is called irreducible if for any $k\in I$, $\Sigma_{i\in I}A_i\neq\Sigma_{i\in I,i\neq k}A_i$. If $\alpha=\Sigma_{i\in I}A_i$ is irreducible, we define

$$|\alpha|=\{A_i|i\in I\} \text{ and } \|\alpha\|=|I|$$

$\|\alpha\|$ is called the order of α.

Proposition 2 Let S be the sub-algebra of EM. $\alpha\in S$, if $\alpha=A$ and $A\subseteq M$, $|A|=0$ or $|A|=1$, then α is the element of any base of S.

Proof Since $|A|=0$ or $|A|=1$

Case 1) $|A|=0$, i.e.$A=\varnothing$

Case 2) $|A|=1$, we can suppose $A=\{n\}$

$\varnothing$ and $\{n\}$ are all moleculars in EM. And molecular[11] must be base. So $\varnothing$ and $\{n\}$ must be the element of any base of S.

Proposition 3 Let S be any sub-algebra of EM. If $S=\{\alpha_1, \alpha_2\}$, then $S=\{\alpha_1, \alpha_2\}$ is a base of S.

Proof Since $(\{\alpha_1\})_{EI}=\{\alpha_1\}$, hence $\alpha_2\notin(\{\alpha_1\})_{EI}$. Similarly $\alpha_1\notin(\{\alpha_2\})_{EI}$. Therefore α_1, α_2 are EI independent. And $S=\{\alpha_1, \alpha_2\}=(\{\alpha_1, \alpha_2\})_{EI}$, so by the definition, we can see α_1, α_2 is the base of S.

Example 1 Let α_1={1}, α_2={1}∨{2}.
Since {1}∨{1}={1}, {1}∧{1}={1}, [{1}∨{2}]∨ [{1}∨{2}]={1}∨{2}, [{1}∨{2}]∧[{1}∨{2}]={1}∨{2}, so{{1}, {1}∨{2}}is sub-algebra, and its base is {1},{1}∨{2} and ({1}, {1}∨{2})$_{EI}$={{1}, {1}∨{2}}.

Proposition4 The base of the sub-algebra of EM isn't unique.
Proof By the following examples, we can prove this proposition.
Example2 S={α_1, α_2, ..., α_{13}}. Where α_1={1}, α_2={1}∨{2}, α_3={3}∨{4}, α_4={2}∨{3}∨{4}, α_5={1}∨{2}∨{3}∨{4}, α_6={1}∨{3}∨{4}, α_7={1, 3}∨{1, 4}, α_8={1, 2}∨{1, 3}∨{1, 4}, α_9={1, 3}∨{1,4}∨{2, 3}∨{2, 4}, α_{10}={3}∨{4}∨{1, 2}, α_{11}={1,3}∨{1, 4}∨{2}, α_{12}={1}∨{2, 3}∨{2, 4}, α_{13}={1, 2}∨{1, 3}∨{1, 4}∨{2, 3}∨{2, 4}.
We can verify that:
(1) S=({1}, {1}∨{2}, {3}∨{4}, {2}∨{3}∨{4})$_{EI}$ and {1}, {1}∨{2}, {3}∨{4}, {2}∨{3}∨{4} is a base of S.
(2) S=({1}, {1, 3}∨{1, 4}∨{2}, {3}∨{4})$_{EI}$ and {1}, {1, 3}∨{1, 4}∨{2}, {3}∨{4}is also a base of the EI algebra of S. And the elements of bases in (1) and (2) are different.

Proposition5 The smallest bases are not unique.
Proof By the following examples, we can prove this proposition.
Example3 Letα_1={{1}, {1}∨{2}, {2}∨{3}, {3}∨{4}}
α_2={{1}, {1,3}∨{2}, {2,4}∨{3}, {3}∨{4}}
since [{1, 3}∨{2}]∨ [{2, 4}∨{3}]={2}∨{3}, {1}∨ [{1, 3}∨{2}]={1}∨{2}, then $(\alpha_1)_{EI}=(\alpha_2)_{EI}$

Definition 5 Let $\alpha=\Sigma_{i\in I}A_i\in EM$, we make the definition as $\overline{\alpha}=\bigcup_{i\in I}A_i$.

Proposition 6 S={α_1, α_2, ..., α_n}, if $\alpha_{n_1},\alpha_{n_2},\cdots,\alpha_{n_i}$ can generate S, then

$$\bigcup_{\alpha\in\{\alpha_{n_1},\alpha_{n_2},\cdots\alpha_{n_i}\}}\overline{\alpha} = \bigcup_{\alpha\in\{a_1,a_2,\cdots a_n\}}\overline{\alpha}$$

Proof Since{ $\alpha_{n_1},\alpha_{n_2},\cdots,\alpha_{n_i}$ } ⊆S, hence $\bigcup_{\alpha\in\{\alpha_{n_1},\alpha_{n_2},\cdots\alpha_{n_i}\}}\overline{\alpha} = \bigcup_{\alpha\in\{a_1,a_2,\cdots a_n\}}\overline{\alpha}$
now suppose ∃ $\beta\in\bigcup_{\alpha\in\{a_1,a_2,\cdots a_n\}}\overline{\alpha}$, s.t. $\beta\notin\bigcup_{\alpha\in\{\alpha_{n_1},\alpha_{n_2},\cdots\alpha_{n_i}\}}\overline{\alpha}$, then β can't be generated by $\alpha_{n_1},\alpha_{n_2},\cdots,\alpha_{n_i}$, since { $\alpha_{n_1},\alpha_{n_2},\cdots,\alpha_{n_i}$ } is the base of S, so they can generate any element of S. They are contradict. So $\forall\beta\in\bigcup_{\alpha\in\{a_1,a_2,\cdots a_n\}}\overline{\alpha}$,it must have $\beta\in\bigcup_{\alpha\in\{\alpha_{n_1},\alpha_{n_2},\cdots\alpha_{n_i}\}}\overline{\alpha}$.

Proposition7 Let $S=\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, let $\alpha_i=\sum_{j\in I_i} A_{ij}$, $i=1, 2, \ldots, n$, if α_i can be generated by

$\alpha_{n_1}, \alpha_{n_2}, \cdots, \alpha_{n_l}$, then $\forall j\in I_j$,

$$A_{ij} = \bigcup_{k\in\{n_1,n_2,\cdots n_l\}} C_k$$

Where $C_k=\phi$ or $C_k=A_{kq}$, $q\in I_k$, $k=n_1,n_2,\ldots,n_l$.

Proof (i) if $\alpha_i=\vee_{n_k\in T}\,\alpha_{n_k}$ ($T=\{n_1,n_2,\ldots,n_l\}$), it is obvious.

(ii) if $\alpha_i=\wedge_{n_k\in T}\,\alpha_{n_k}$ ($T=\{n_1,n_2,\ldots,n_l\}$)$=\sum_{j\in I_i} A_{ij}$, then

$$\alpha_i=\Big(\sum_{j_1\in I_{n_1}} A_{n_1 j_1}\Big)\wedge\Big(\sum_{j_2\in I_{n_2}} A_{n_2 j_2}\Big)\wedge\ldots\wedge\Big(\sum_{j_m\in I_{n_m}} A_{n_m j_m}\Big), m\in\{1,2,\ldots,l\}$$

$$= \sum_{j\in I_{n_1}, j_2\in I_{n_2},\cdots j_m\in I_{n_m}} A_{n_1 j_1}\cup A_{n_2 j_2}\cup\cdots\cup A_{n_m j_m}=\sum\bigcup_{k\in T} C_k$$

so we have $A_{ij} = \bigcup_{k\in\{n_1,n_2,\cdots n_l\}} C_k$ ($\forall j\in I_j$).

Note 1 The proposition 7 can be used to judge EI independent. For example:

Example 4 Let $\alpha_1=\{2,4,5\}\vee\{1,2\}$, $\alpha_2=\{4\}\vee\{1,2\}$, $\alpha_3=\{3,4\}\vee\{1\}$,

$\alpha_4=\{1,4\}\vee\{2\}\vee\{3\}$, $\alpha_5=\{1,4\}\vee\{1,2\}\vee\{3,4\}$.

Knowing from Proposition 7

$$\alpha_3\notin(\alpha_1, \alpha_2)_{EI},\ \alpha_2\notin(\alpha_1, \alpha_3)_{EI},\ \alpha_1\notin(\alpha_2, \alpha_3)_{EI}$$

so $\alpha_1, \alpha_2, \alpha_3$ is EI independent. For the same reason $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is EI independent.

Note 2 The converse proposition of Proposition 7 doesn't found. The converse proposition is : for $S=\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, let $\alpha_i=\sum_{j\in I_i} A_{ij}$, $i=1, 2, \ldots, n$, if $\forall j\in I_j$,

$$A_{ij} = \bigcup_{1\le k\le n} C_k$$

then α_i can be generated by $\alpha_{n_1}, \alpha_{n_2}, \cdots, \alpha_{n_l}$.

Example 5 Let $\alpha_1=\{1\}\vee\{2,3\}$, $\alpha_2=\{3, 4\}\vee\{1, 2\}$, $\alpha_3=\alpha_1\vee\alpha_2=\{1\}\vee\{2, 3\}\vee\{3, 4\}$,

$\alpha_4=\alpha_1\wedge\alpha_2=\{1, 3, 4\}\vee\{1, 2\}\vee\{2, 3, 4\}$, $\alpha_5=\{1\}$, $\alpha_6=\{2, 3\}\vee\{3, 4\}$

$\alpha_3, \alpha_4, \alpha_5, \alpha_6$ can be generated by α_1, α_2 using the method of proposition7, but α_3, α_4 can be generated by α_1, α_2 and α_5, α_6 can't be generated by α_1, α_2.

Proposition8 $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}\subseteq EM$, $\alpha_i\subseteq M$, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are EI independent if

$\|\alpha_1\vee\alpha_2\vee\ldots\vee\alpha_n\|=\|\alpha_1\|+\|\alpha_2\|+\ldots+\|\alpha_n\|$, (i.e. $\alpha_1\vee\alpha_2\vee\ldots\vee\alpha_n$ is irreducible).

Proof Since $\|\alpha_1\vee\alpha_2\vee\ldots\vee\alpha_n\|=\|\alpha_1\|+\|\alpha_2\|+\ldots+\|\alpha_n\|$, though proposition 8 we can know that: $\forall$ i, $i\in\{1, 2, \ldots, n\}$, $\alpha_i\notin(\{\alpha_j|\alpha_j\neq\alpha_i, j\in\{1, 2, \ldots, n\}\})_{EI}$. So $\alpha_1, \alpha_2, \ldots, \alpha_n$ are EI independent

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