# Notes on the decision theorems that the equation type Ⅱ of a fuzzy matrix has the solutions when the index is one

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**Abstract:** In this paper, the decision methods that the equation type Ⅱ of a fuzzy matrix has the solutions when the index is one were again studied with the both concepts of the linear dependence of the row (column) vectors of this fuzzy matrix and the maximum row (column) vector of this fuzzy matrix, and we had obtained the simple and clear decision theorems.

**Keywords:** fuzzy matrix, linear dependence of the row (column) vectors of a fuzzy matrix, equation type Ⅱ of a fuzzy matrix, maximum row (column) vector of a fuzzy matrix.

## 1 The definitions

**Definition** 1 [1] Let $B \in L^{n\times n}$ is a non-zero fuzzy matrix. An equation in the form of

$$B_{n\times n} = X_{n\times t}X^T_{t\times n}$$

is called a fuzzy relational non-deterministic equation of type Ⅱ of B, or an equation type Ⅱ of B. The t is called its index. The $X_{n\times t}$ such that this equation holds is called a solution of the equation when the index is t.

**Definition** 2. Let $B=(b_{ij})\in L^{n\times n}$ is a fuzzy matrix. The k-th column vector of B is called the maximum column vector of B, if

$$\max\{b_{11}, b_{12}, \cdots, b_{1n}\} = b_{1k}, k\in\{1,2,\cdots,n\}$$

$$\cdots\cdots\cdots$$

$$\max\{b_{n1}, b_{n2}, \cdots, b_{nn}\} = b_{nk}.$$

And the h-th row vector of B is called the maximum row vector of B, if

$$\max\{b_{11}, b_{21}, \cdots, b_{n1}\} = b_{h1}, h\in\{1,2,\cdots,n\}$$

$$\cdots\cdots\cdots$$

$$\max\{b_{1n}, b_{2n}, \cdots, b_{nn}\} = b_{hn}.$$

In this paper the other definitions and symbols see [2].

## 2 The decision theorems that the equation type Ⅱ of a fuzzy matrix has the solutions when the index is one

**Theorem** 1 Let B is a symmetrical diagonally dominant fuzzy matrix. The equation type Ⅱ of B has the solutions when the index is one, if and only if B has the unique linearly independent row vector, and B has the unique linearly independent column vector.

**Proof.** Necessity. Since the equation type Ⅱ of B has the solutions when the index is one, let $B=(b_{ij})\in L^{n\times n}$, and let

$$\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{1n} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_1 \\ \cdots \\ a_n \end{bmatrix} (c_1, \cdots, c_n) = \begin{bmatrix} a_1(c_1, \cdots, c_n) \\ \cdots\cdots\cdots \\ a_n(c_1, \cdots, c_n) \end{bmatrix}.$$

If $a_k = \max\{a_1, \cdots, a_n\}$, then the row vectors of B are that

$$\begin{aligned} &\text{the } 1-\text{st row} \quad a_1(c_1, \cdots, c_n) = a_1 a_k(c_1, \cdots, c_n), \\ &\cdots\cdots \qquad \cdots\cdots \\ &\text{the } k-\text{th row} \quad a_k(c_1, \cdots, c_n), \\ &\cdots\cdots \qquad \cdots\cdots \\ &\text{the } n-\text{th row} \quad a_n(c_1, \cdots, c_n) = a_n a_k(c_1 \cdots c_n). \end{aligned}$$

Because B is a non-zero matrix, thus $a_k(c_1 \cdots c_n)$ is a non-zero vector, therefore $a_k(c_1 \cdots c_n)$ is the unique linearly independent row vector of B.

To prove analogously that B has the unigue linearly independent column vector.

Sufficiency. Suppose that $(b_{1i} \cdots b_{ii} \cdots b_{in})(1 \leqslant i \leqslant n)$ is the unique linearly independent row vector of B, then there is $k_1, \cdots, k_n \in L$, such that

$$(b_{1h} \cdots b_{hh} \cdots b_{hn}) = k_h(b_{1i} \cdots b_{ii} \cdots b_{in}), (1 \leqslant h \leqslant n), \tag{1}$$

thus

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{1n} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} k_1(b_{1i} \cdots b_{ii} \cdots b_{in}) \\ \cdots\cdots\cdots\cdots \\ k_n(b_{1i} \cdots b_{ii} \cdots b_{in}) \end{bmatrix}$$

$$= \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} (b_{1i} \cdots b_{ii} \cdots b_{in}) = \left[ \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} b_{1i}, \cdots, \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} b_{in} \right] \tag{2}$$

If $b = \max\{b_{1i}, \cdots, b_{ii}, \cdots, b_{in}\}$, then the i-th row of B is $b(b_{1i}, \cdots, b_{ii}, \cdots, b_{in})$. and the i-th column of B is $b(b_{1i} \cdots b_{ii} \cdots b_{in})^T$. By the expression (2) we may see that the all column of B are that

$$\left.\begin{aligned} &b_{1i}(k_1 \cdots, k_n)^T = b_{1i} b(k_1, \cdots, k_n)^T \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &b_{ii}(k_1 \cdots, k_n)^T = b_{ii} b(k_1, \cdots, k_n)^T \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &b_{in}(k_1 \cdots, k_n)^T = b_{in} b(k_1, \cdots, k_n)^T \end{aligned}\right\} \tag{3}$$

Therefore the every columns of B are a linear combination of the column $b\ (k_1, \cdots, k_n)^T$ of B. Because B is a non-zero matrix, thus $b\ (k_1, \cdots, k_n)^T$ is a non-zero vector, too. So $b(k_1, \cdots, k_n)^T$ is the unique column vector of the linear independence of B.

By the expressions both of (2) and (3) we obtain that

$$B = \left[ \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} b_{1i}, \cdots, \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} b_{in} \right] = \left[ \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} bb_{1i}, \cdots, \begin{bmatrix} k_1 \\ \cdots \\ k_n \end{bmatrix} bb_{in} \right] = \left[ \begin{bmatrix} k_1 b \\ \cdots \\ k_n b \end{bmatrix} b_{1i}, \cdots, \begin{bmatrix} k_1 b \\ \cdots \\ k_n b \end{bmatrix} b_{in} \right]$$

$$= \left[ \begin{bmatrix} b_{1i} \\ \cdots \\ b_{in} \end{bmatrix} b_{1i}, \cdots, \begin{bmatrix} b_{1i} \\ \cdots \\ b_{in} \end{bmatrix} b_{in} \right] = \begin{bmatrix} b_{1i} \\ \cdots \\ b_{in} \end{bmatrix} (b_{1i} \cdots b_{in}) = CC^T,$$

where $C = (b_{1i}, \cdots, b_{in})^T$. So the equation type Ⅱ of B has the solutions when the index is one.

**Theorem** 2 . Let B is a non − zero symmetrical diagonally dominant fuzzy matrix. The equation type Ⅱ of B has the solutions when the index is one , if and only if B is equal to composition of the both of the unique linearly independent column vector of B and the unique linearly independent row vector of B.

**Theorem** 3. **(The first decision theorem that the equation type Ⅱ of a fuzzy matrix has the solutions when the index is one)** Let B is a symmetrical diagonally dominant fuzzy matrix. The equation type Ⅱ of B has the solutions when the index is one, if and only if B is egual to a composition of the both of the maximum column veceor of B and the maximum row veceor of B.

**Theorem** 4 [1] **(The second decition theorem that the equation type Ⅱ of a fuzzy matrix has the solutions when the index is one)** Let $B = (b_{ij}) \in L^{n \times n}$ is a non-zero symmetrical diagonally dominant fuzzy matrix. The equation type Ⅱ of B has the solutions when the index is one, if and only if

$$(b_{11} \quad b_{12} \quad \cdots \quad b_{nn})^T$$

is the unique solution of the equation.

**Theorem** 5 Let B is a non-zero symmetrical diagonally dominant fuzzy matrix. If the equation type Ⅱ of B has the solutions when the index is one, then the maximum column (row) of B is composed of the main diagonal elements of B.

By Theorem 1 we have obtained direct Theorem 6, too.

**Theorem** 6. let B is a non-zero symmetrical diagonally dominant fuzzy matrix. The equation type Ⅱ of B has the solutions when the index is one

(1) if and only if $\rho_r(B) = 1$, and at the same time $\rho_c(B) = 1$;

(2) if and only if $\rho_s(B) = 1$;

(3) if and only if $\rho(B) = 1$.

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# Equation type Ⅲ of a fuzzy matrix

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**Abstract.** A definition of the equation type Ⅲ of a fuzzy matrix was given in this paper, and the both of the properties and a solution for the equation were discussed preliminaryly. And we have been obtained a decision theorem that the equation type Ⅲ of a fuzzy matrix has the solutions.

**Keywords.** fuzzy matrix, sub − transposed fuzzy matrix, sub − symmetric fuzzy matrix, equation type Ⅲ of a fuzzy matrix

## 1. foundation

**Definition** 1 [1]. Let $A = (a_{ij}) \in L^{n\times m}$. And $A^{ST} = (a_{kh}^{ST}) \in L^{m\times n}$ is called a sub − transposed fuzzy matrix of A, if

$$a_{kh}^{ST} = a_{n-h+1,m-k+1}; i, h = 1, 2, \cdots, n. \ j, k = 1, 2, \cdots, m.$$

**Definition** 2 [1]. Let $A = (a_{ij}) \in L^{n\times n}$. And A is called a sub − symmetric fuzzy matrix, if $A = A^{ST}$.

The both of the concepts and the signs, which aren't particularly pointed out in this paper, will be found in references[1~2].

## 2. The Concept of the equation type Ⅲ of a fuzzy matrix

**Definition** 3. Let $B \in L^{n\times n}$. An equation in the form of

$$B = X_{nxt}(X^{ST})_{t\times n} \qquad (2,1)$$

is called a fuzzy relation indefinite equation type Ⅲ of the B or the equation type Ⅲ of B. (Where B is known, and X is unknown). The t is called its index, and the fuzzy matrix $X_{nxt}$ satified expression (2, 1) is called the solution matrix of the equation when the index is t. If the equation type Ⅲ of B has the solutions when the index is some numbers, then we generally may speake that the equation type Ⅲ of B has the solutions.

**Theorem** 1. If the equation type Ⅲ of $B \in L^{n\times n}$ has the solutions when the index is p, then the equation type Ⅲ of B has the solutions when the index is p + 2k, where k is an arbitrary positive integer.

**Proof.** Because the equation type Ⅲ of B has the solutions when the index is p, then there is $C \in L^{n\times p}$ such that

$$B = C\,C^{ST}.$$

We take $A = (O \quad C \quad O)$, where $O \in L^{n\times k}$ is a zero matrix, then

$$A^{ST} = \begin{bmatrix} O^{ST} \\ C^{ST} \\ O^{ST} \end{bmatrix},$$

by reason of

$$A\,A^{ST}=(O\quad C\quad O)\begin{bmatrix}O^{ST}\\ C^{ST}\\ O^{ST}\end{bmatrix}=O+C\,C^{ST}+O=B.$$

And notes that $A\in L^{n\times(P+2k)}$, therefore A is a solution of the equation type Ⅲ of B when the index is $p+2k$.

**Theorem** 2. If the equation type Ⅲ of $B_i\in L^{n\times n}$ has the solutions when the index is $p_i$, where $i=1,2$, then the equation type Ⅲ of $(B_1+B_2)$ has the solution when the index is $2p_1+p_2$(or $2p_2+p_1$).

**Proof.** Because the equation type Ⅲ of $B_1(B_2)$ has the solutions when the index is $p_1(p_2)$, then there is $C_1\in L^{n\times p_1}$ and $C_2\in L^{n\times p_2}$, respectively, such that

$$B_1=C_1\,C_1^{ST}\qquad\text{and}\qquad B_2=C_2\,C_2^{ST}.$$

We take that

$$A=(C_1\quad C_2\quad C_1)\qquad\text{and}\qquad D=(C_2\quad C_1\quad C_2),$$

then

$$A^{ST}=\begin{bmatrix}C_1^{ST}\\ C_2^{ST}\\ C_1^{ST}\end{bmatrix}\qquad\text{and}\qquad D^{ST}=\begin{bmatrix}C_2^{ST}\\ C_1^{ST}\\ C_2^{ST}\end{bmatrix},$$

and so

$$A\,A^{ST}=B_1+B_2\qquad\text{and}\qquad D\,D^{ST}=B_1+B_2.$$

And note that $A\in L^{n\times(2P_1+P_2)}$ and $D\in L^{n\times(2P_2+P_1)}$, therefore the equation type Ⅲ of $(B_1+B_2)$ has the solution when the index is $2p_1+p_2$ or $2p_2+p_1$.

To prove analogously that

**Theorem** 3. Let the equation type Ⅲ of $B_i\in L^{n\times n}$ has the solutions when the index is $p_i$, where $i=1,2,\cdots,k$, then the equation type Ⅲ of $(B_1+B_2+\cdots+B_k)$ has the solutions when the index is

$$\min\{2p_2+2p_3+\cdots+2p_k+p_1,\,2p_1+2p_3+\cdots+2p_k+p_2,\\ \cdots,\,2p_1+\cdots+2p_{k-1}+p_k\},$$

where k is an arbitrary positive integer.

**Theorem** 4. If the equation type Ⅲ of $B\in L^{n\times n}$ has the solutions when the index is p, then the equation type Ⅲ of $\lambda B$ too has the solutions when the index is p, where $\lambda\in(0,1]$.

**Proof.** Let $\quad B=A\,A^{ST}\quad$ and $\quad A\in L^{n\times p}$

since

$$(\lambda A)\,(\lambda A)^{ST}=\lambda(A\,A^{ST})=\lambda B,$$

and $\lambda A\in L^{n\times p}$, therefore the equation type Ⅲ of $\lambda B$ has the solutions when the index is p.

Proceed to the next step we have that

**Theorem** 5. If the equation type Ⅲ of $B_i$ has the solutions, where $i=1,\cdots,k$, then the equation type Ⅲ of $(\lambda_1B_1+\lambda_2B_2+\cdots+\lambda_kB_k)$ has solutions, too.

**Theorem** 6. Let $B\in L^{n\times n}$. If the equation type Ⅲ of B has not any solution when the index is p ($p\geqslant 2$), then the equation type Ⅲ of B has not any solution when the index $t\leqslant p-2k$, where k is a positive integer and $p-2k\geqslant 1$.

In fact, suppose the equation type Ⅲ of B has the solutions when the index $t = p - 2h(p - 2h \geqslant 1)$, then by Theorem 1 knows that the equation type Ⅲ of B has the solutions when the index $t = (p - 2h) + 2h = p$, and this is a contradiction. Therefore this theorem is right.

**Theorem 7. (The decision theorem that the equation type Ⅲ of a fuzzy matrix has the solutions)** A equation type Ⅲ of a fuzzy matrix has a solution if and only if B is a sub - symmetric fuzzy square matrix.

**Proof.** Necessity. Because the equation type Ⅲ of B has a solution, we may assume without losing generality that the equation type Ⅲ of B has the solutions when the index $t = p$, then there is $A \in L^{n \times p}$ such that

$$B = A\ A^{ST},$$

and so

$$B^{ST} = (A\ A^{ST})^{ST} = (A^{ST})^{ST}\ A^{ST} = A \circ A^{ST} = B.$$

Therefore B is a sub - symmetric fuzzy square matrix.

Sufficiency. Suppose that $B = (b_{ij}) \in L^{n \times n}$ and $B = B^{ST}$. Assume that an element $b_{ij}$ in B $(i, j = 1, 2, \cdots, n;$ and $i + j \neq n + 1)$ is an element of a twin - element type of B. Presume that an element $b_{ij}$ of B never varies, and other elements of B can be changed into zero. And we obtain a fuzzy matrix $B_{ij}$:

$$B_{ij} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & & \cdots & & \cdots & & \cdots \\ \cdots & & b_{ij} & \cdots & 0 & & \cdots \\ \cdots & & \cdots & & \cdots & & \cdots \\ \cdots & & 0 & \cdots & b_{ij} & & \cdots \\ \cdots & & \cdots & & \cdots & & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \quad \begin{pmatrix} i, j = 1, 2, \cdots, n \\ \text{and } i + j \neq n + 1 \end{pmatrix}$$

If

$$A_{ij} = \begin{pmatrix} 0 & \cdots & 0 & b_{ij} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b_{ij} & 0 & \cdots & 0 \end{pmatrix}^{T}, (i + j \neq n + 1)$$

then

$$A_{ij}^{ST} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b_{ij} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{ij} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, (i + 1 \neq n + 1)$$

therefore

$$A_{ij} \circ A_{ij}^{ST} = B_{ij},$$

and so the equation type Ⅲ of $B_{ij}$ has the solutions (where $i, j = 1, 2, \cdots, n$; and $i + j \neq n + 1$).

Suppose that an element $b_{kh}$ in B (where $k, h = 1, 2, \cdots, n$; and $k + h = n + 1$) is an element of a single - element type of B. Presume that an element $b_{kh}$ of B never varies, and other elements of B can be changed into zero. And we obtain a fuzzy matrix $B_{kh}$:

$$B_{kh} = \begin{bmatrix} 0 & \cdots & 0 \\ \cdots & \phantom{x} \cdot^{\cdot^{\cdot}} & \cdots \\ \cdots & b_{kh} & \cdots \\ \cdots \ \cdot^{\cdot^{\cdot}} & & \cdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{pmatrix} k, h = 1, 2, \cdots, n \\ \text{and } k + h = n + 1 \end{pmatrix}$$

If

$$A_{kh} = (0 \quad \cdots \quad 0 \quad b_{kh} \quad 0 \quad \cdots \quad 0)^T,$$

then

$$A_{kh}^{ST} = (0 \quad \cdots \quad 0 \quad b_{kh} \quad 0 \quad \cdots \quad 0),$$

therefore

$$A_{kh} \circ A_{kh}^{ST} = B_{kh}.$$

And so the equation type Ⅲ of $B_{kh}$ has the solutions, too. (where $k, h = 1, 2, \cdots, n$; and $k + h = n + 1$). Because

$$B = B_{11} + B_{12} + \cdots + B_{1n} + B_{21} + B_{22} + \cdots + B_{2n-1} + \cdots + + B_{n-11} + B_{n-12} + B_{n1}$$

Therefore the equation type Ⅲ of B has the solutions by Theorem 3.

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