Decompositions of Intuitionistic Fuzzy Sets

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Abstract
Based on the forms of upper cut set and lower cut set of intuitionistic fuzzy set, the decomposition problems of the intuitionistic fuzzy set are discussed in this paper. And two types of the decompositions are given.
Keywords: Intuitionistic fuzzy set, cut sets, upper decompositions, lower decompositions.

1. Preparation

Let us take the following set:

$$D=\{(x,y)\in[0,1]\times[0,1] \mid x+y\le 1\}$$

First we introduce operations on D as follows:

Definition 1.1 For every $(a_t,b_t)\in D, t\in T$, we define:

$$\bigvee_{t\in T}(a_t,b_t)=(\bigvee_{t\in T}a_t,\bigwedge_{t\in T}b_t);$$
$$\bigwedge_{t\in T}(a_t,b_t)=(\bigwedge_{t\in T}a_t,\bigvee_{t\in T}b_t);$$
$$(a_t,b_t)'=(b_t,a_t)$$

Definition 1.2 For each $(a_i,b_i)\in D, i=1,2$. We define:

$$(a_1,b_1)=(a_2,b_2) \quad \textit{iff} \quad a_1=a_2 \ \& \ b_1=b_2;$$
$$(a_1,b_1)\le(a_2,b_2) \quad \textit{iff} \quad a_1\le a_2 \ \& \ b_1\ge b_2;$$
$$(a_1,b_1)<(a_2,b_2) \quad \textit{iff} \quad (a_1,b_1)\le(a_2,b_2) \ \& \ (a_1,b_1)\ne(a_2,b_2);$$
$$(a_1,b_1)\dot{<}(a_2,b_2) \quad \textit{iff} \quad a_1<a_2 \ \& \ b_1>b_2.$$

It is easy to prove the following results.

Theorem 1.1 Let $\alpha,\alpha_t\in D, t\in T$, then

(1) $\alpha\wedge(\bigvee_{t\in T}\alpha_t)=\bigvee_{t\in T}(\alpha\wedge\alpha_t)$;

(2) $\alpha\vee(\bigwedge_{t\in T}\alpha_t)=\bigwedge_{t\in T}(\alpha\vee\alpha_t)$.

Theorem 1.2 The system $(D,\le,\wedge,\vee)$ is a complete lattice with the order-reversing involution" $'$ ". And it has maximal element $\tilde{1}=(1,0)$ and minimal element $\tilde{0}=(0,1)$.

Let a set X be fixed. An intuitionistic fuzzy set (IFS) A in X is an object of the following form (see [1]):

$$A=\{<x,\mu_A(x),\nu_A(x)>|\,x\in E\}$$

where the functions $\mu_A:E\to[0,1]$ and $\nu_A:E\to[0,1]$ define the degree of membership and the degree of non-membership of the element $x\in E$, respectively, and for every $x\in E$,

$$0\le\mu_A(x)+\nu_A(x)\le 1$$

For ordinary fuzzy set A, its form is $A=\{<x,\mu_A(x),1-\mu_A(x)>|\,x\in X\}$.

For IFSs A, B and A_t $(t\in T)$, the following definitions are valid:

$$A\subseteq B \quad \textit{iff} \quad (\forall x\in E)(\mu_A(x)\le\mu_B(x)\ \&\ \nu_A(x)\ge\nu_B(x));$$

$$A\supseteq B \quad \textit{iff} \quad B\subseteq A;$$

$$A=B \quad \textit{iff} \quad (\forall x\in E)(\mu_A(x)=\mu_B(x)\ \&\ \nu_A(x)=\nu_B(x));$$

$$\overline{A}=\{<x,\nu_A(x),\mu_A(x)>|\,x\in X\};$$

$$\bigcap_{t\in T}A_t=\{<x,\bigwedge_{t\in T}\mu_{A_t}(x),\bigvee_{t\in T}\nu_{A_t}(x)>|\,x\in X\};$$

$$\bigcup_{t\in T}A_t=\{<x,\bigvee_{t\in T}\mu_{A_t}(x),\bigwedge_{t\in T}\nu_{A_t}(x)>|\,x\in X\};$$

$$Q_{\alpha,\beta}(A)=\{<x,\alpha\wedge\mu_A(x),\beta\vee\nu_A(x)>|\,x\in X\},\quad((\alpha,\beta)\in D).$$

Definition 1.3 Let A be an IFS in X, $(\lambda_1,\lambda_2)\in D$, we call the following sets

$$A_{(\lambda_1,\lambda_2)}=\{x\in X:\mu_A(x)\ge\lambda_1\ \&\ \nu_A(x)\le\lambda_2\},$$

$$A_{(\lambda_1,\lambda_2)\cdot}=\{x\in X:\mu_A(x)>\lambda_1\ \&\ \nu_A(x)<\lambda_2\},$$

$$A_{(\lambda_1\cdot,\lambda_2)}=\{x\in X:\mu_A(x)>\lambda_1\ \&\ \nu_A(x)\le\lambda_2\},$$

$$A_{(\lambda_1,\lambda_2\cdot)}=\{x\in X:\mu_A(x)\ge\lambda_1\ \&\ \nu_A(x)<\lambda_2\}$$

(λ_1,λ_2)-upper cut set, (λ_1,λ_2)-strong upper cut set, $(\lambda_1\cdot,\lambda_2)$-upper cut set and $(\lambda_1,\lambda_2\cdot)$-upper cut set, respectively.

Obviously,

$$A_{(\lambda_1,\lambda_2)\cdot}\subseteq\begin{matrix}A_{(\lambda_1\cdot,\lambda_2)}\\ A_{(\lambda_1,\lambda_2\cdot)}\end{matrix}\subseteq A_{(\lambda_1,\lambda_2)},\qquad A_{(0,1)}=X,\quad A_{(1,0)\cdot}=\phi$$

Definition 1.4 Let A be an IFS in X, $(\lambda_1,\lambda_2)\in D$, we call the following sets

$$A^{(\lambda_1,\lambda_2)}=\{x\in X:\mu_A(x)\le\lambda_1\ \&\ \nu_A(x)\ge\lambda_2\},$$

$$A^{(\lambda_1,\lambda_2)\cdot}=\{x\in X:\mu_A(x)<\lambda_1\ \&\ \nu_A(x)>\lambda_2\},$$

$$A^{(\lambda_1\cdot,\lambda_2)} = \{x \in X : \mu_A(x) < \lambda_1 \ \&\ \nu_A(x) \geq \lambda_2\},$$
$$A^{(\lambda_1,\lambda_2\cdot)} = \{x \in X : \mu_A(x) \leq \lambda_1 \ \&\ \nu_A(x) > \lambda_2\}$$

(λ_1,λ_2)-lower cut set, (λ_1,λ_2)-strong lower cut set, $(\lambda_1\cdot,\lambda_2)$-lower cut set and $(\lambda_1,\lambda_2\cdot)$-lower cut set, respectively.

Obviously, $A^{(\lambda_1,\lambda_2)\cdot} \subseteq \begin{matrix} A^{(\lambda_1\cdot,\lambda_2)} \\ A^{(\lambda_1,\lambda_2\cdot)} \end{matrix} \subseteq A^{(\lambda_1,\lambda_2)}$.

2. Decompositions of IFS

Let B be a classical subset of X, $(\lambda_1,\lambda_2) \in D$, we define an IFS in X as follows:

$$R_{\lambda_1,\lambda_2}(B) = \begin{cases} \{< x, \lambda_1, \lambda_2 > \mid x \in X\}, & x \in B \\ \{< x, 1, 0 > \mid x \in X\}, & x \notin B \end{cases}$$

It is easy to prove the following proposition.

Proposition 2.1 Let A, B be two IFSs in X, E, F be two classical subsets of X, $(\lambda_1,\lambda_2), (\lambda_3,\lambda_4) \in D$, we have

(1) *If* $(\lambda_1,\lambda_2) \leq (\lambda_3,\lambda_4)$, *then* $Q_{\lambda_1,\lambda_2}(A) \subseteq Q_{\lambda_3,\lambda_4}(A)$, $R_{\lambda_1,\lambda_2}(E) \subseteq R_{\lambda_3,\lambda_4}(E)$;

(2) *If* $A \subseteq B, E \subseteq F$, *then* $Q_{\lambda_1,\lambda_2}(A) \subseteq Q_{\lambda_1,\lambda_2}(B)$, $R_{\lambda_1,\lambda_2}(E) \subseteq R_{\lambda_1,\lambda_2}(F)$.

Theorem 2.1 let A be an IFS in X, then

(1) (*upper decomposition I*) $A = \bigcup_{(\lambda_1,\lambda_2)\in D} Q_{\lambda_1,\lambda_2}(A_{(\lambda_1,\lambda_2)})$;

(2) (*upper decomposition II*) $A = \bigcup_{(\lambda_1,\lambda_2)\in D} Q_{\lambda_1,\lambda_2}(A_{(\lambda_1,\lambda_2)\cdot})$;

(3) (*upper decomposition III*) $A = \bigcup_{(\lambda_1,\lambda_2)\in D} Q_{\lambda_1,\lambda_2}(A_{(\lambda_1\cdot,\lambda_2)})$;

(4) (*upper decomposition IV*) $A = \bigcup_{(\lambda_1,\lambda_2)\in D} Q_{\lambda_1,\lambda_2}(A_{(\lambda_1,\lambda_2\cdot)})$.

Proof. We only prove (1), and others are similar.

$$\bigcup_{(\lambda_1,\lambda_2)\in D} Q_{\lambda_1,\lambda_2}(A_{(\lambda_1,\lambda_2)}) = \bigcup_{(\lambda_1,\lambda_2)\in D} \{< x, \lambda_1 \wedge \mu_{A_{(\lambda_1,\lambda_2)}}(x), \lambda_2 \vee \nu_{A_{(\lambda_1,\lambda_2)}}(x) > \mid x \in X\}$$
$$= \bigcup_{\mu_A(x)\geq\lambda_1, \nu_A(x)\leq\lambda_2} \{< x, \lambda_1, \lambda_2 > \mid x \in X\} = \{< x, \bigvee_{\mu_A(x)\geq\lambda_1} \lambda_1, \bigwedge_{\nu_A(x)\leq\lambda_2} \lambda_2 > \mid x \in X\}$$
$$= \{< x, \mu_A(x), \nu_A(x) > \mid x \in X\} = A..$$

□

From reference (4), we can easily get

Theorem 2.2 (*upper decomposition V*) Let A be an IFS in X, then

$$A = \bigcup_{(\lambda_1,\lambda_2)\in D_Q} Q_{\lambda_1,\lambda_2}(A_{(\lambda_1,\lambda_2)})$$

where $D_Q = \{(\lambda_1,\lambda_2) : \lambda_1 + \lambda_2 \leq 1, \lambda_1, \lambda_2 \in Q\}$, Q denotes the set of all rational numbers in interval $(0,1)$.

Lemma 2.1 Let A be an IFS in X, $(\lambda_{1t},\lambda_{2t})\in D, t\in T$, then

$$A_{(\bigvee_{t\in T}\lambda_{1t},\bigwedge_{t\in T}\lambda_{2t})}=\bigcap_{t\in T}A_{(\lambda_{1t},\lambda_{2t})}$$

The proof of above lemma is straightforward.

Theorem 2.3 (*upper decomposition VI*) Let A be an IFS in X, and the mapping $H: D\to P(X)$ ($P(X)$ denotes the set of all subsets of X), $(\lambda_1,\lambda_2)\mapsto H(\lambda_1,\lambda_2)$ satisfies $A_{(\lambda_1,\lambda_2)\cdot}\subseteq H(\lambda_1,\lambda_2)\subseteq A_{(\lambda_1,\lambda_2)}$ *for all* $(\lambda_1,\lambda_2)\in D$. Then

(1) $A=\bigcup_{(\lambda_1,\lambda_2)\in D}Q_{\lambda_1,\lambda_2}(H(\lambda_1,\lambda_2))$;

(2) *If* $(\lambda_1,\lambda_2)\dot{<}(\lambda_3,\lambda_4)$, *then* $H(\lambda_1,\lambda_2)\supseteq H(\lambda_3,\lambda_4)$;

(3) $A_{(\alpha_1,\alpha_2)}=\bigcap_{(\lambda_1,\lambda_2)\dot{<}(\alpha_1,\alpha_2)}H(\lambda_1,\lambda_2)$, $((\alpha_1,\alpha_2)\neq(0,1))$;

(4) $A_{(\alpha_1,\alpha_2)\cdot}\supseteq\bigcap_{(\lambda_1,\lambda_2)\dot{>}(\alpha_1,\alpha_2)}H(\lambda_1,\lambda_2)$, $((\alpha_1,\alpha_2)\neq(1,0))$.

Proof. (1) From $A_{(\lambda_1,\lambda_2)\cdot}\subseteq H(\lambda_1,\lambda_2)\subseteq A_{(\lambda_1,\lambda_2)}$, proposition (2), and upper decomposition *I*, *II*, we can easily get (1).

(2) $(\lambda_1,\lambda_2)\dot{<}(\lambda_3,\lambda_4)\Rightarrow H(\lambda_1,\lambda_2)\supseteq A_{(\lambda_1,\lambda_2)\cdot}\supseteq A_{(\lambda_3,\lambda_4)}\supseteq H(\lambda_3,\lambda_4)$.

(3) $(\lambda_1,\lambda_2)\dot{<}(\alpha_1,\alpha_2)\Rightarrow H(\lambda_1,\lambda_2)\supseteq A_{(\lambda_1,\lambda_2)\cdot}\supseteq A_{(\alpha_1,\alpha_2)}$

$\Rightarrow\bigcap_{(\lambda_1,\lambda_2)\dot{<}(\alpha_1,\alpha_2)}H(\lambda_1,\lambda_2)\supseteq A_{(\alpha_1,\alpha_2)}$;

On the other hand, from lemma 2.1, we have

$H(\lambda_1,\lambda_2)\subseteq A_{(\lambda_1,\lambda_2)}\Rightarrow\bigcap_{(\lambda_1,\lambda_2)\dot{<}(\alpha_1,\alpha_2)}H(\lambda_1,\lambda_2)\subseteq\bigcap_{(\lambda_1,\lambda_2)\dot{<}(\alpha_1,\alpha_2)}A_{(\lambda_1,\lambda_2)}=A_{(\alpha_1,\alpha_2)}$.

(4) $(\lambda_1,\lambda_2)\dot{>}(\alpha_1,\alpha_2)\Rightarrow A_{(\alpha_1,\alpha_2)\cdot}\supseteq A_{(\lambda_1,\lambda_2)}\supseteq H(\lambda_1,\lambda_2)$

$\Rightarrow A_{(\alpha_1,\alpha_2)\cdot}\supseteq\bigcup_{(\lambda_1,\lambda_2)\dot{>}(\alpha_1,\alpha_2)}H(\lambda_1,\lambda_2)$. □

It is similar to above discussion, we can get anther type of decompositions of IFS.

Theorem 2.4 Let A be an IFS in X, then

(1) (*lower decomposition I*) $A=\bigcap_{(\lambda_1,\lambda_2)\in D}R_{\lambda_1,\lambda_2}(A^{(\lambda_1,\lambda_2)})$;

(2) (*lower decomposition II*) $A=\bigcap_{(\lambda_1,\lambda_2)\in D}R_{\lambda_1,\lambda_2}(A^{(\lambda_1,\lambda_2)\cdot})$;

(3) (*lower decomposition III*) $A=\bigcap_{(\lambda_1,\lambda_2)\in D}R_{\lambda_1,\lambda_2}(A^{(\lambda_1\cdot,\lambda_2)})$;

(4) (*lower decomposition IV*) $A=\bigcap_{(\lambda_1,\lambda_2)\in D}R_{\lambda_1,\lambda_2}(A^{(\lambda_1,\lambda_2\cdot)})$;

(5) (*lower decomposition V*) $A=\bigcap_{(\lambda_1,\lambda_2)\in D_Q}R_{\lambda_1,\lambda_2}(A^{(\lambda_1,\lambda_2)})$.

Theorem 2.5 (*lower decomposition VI*) Let A be an IFS in X, the mapping $H: D\to P(X)$ ($P(X)$ denotes the set of all subsets of X), $(\lambda_1,\lambda_2)\mapsto H(\lambda_1,\lambda_2)$

satisfies $A^{(\lambda_1,\lambda_2)\cdot} \subseteq H(\lambda_1,\lambda_2) \subseteq A^{(\lambda_1,\lambda_2)}$ *for all* $(\lambda_1,\lambda_2) \in D$. Then

(1) $A = \bigcap_{(\lambda_1,\lambda_2)\in D} R_{\lambda_1,\lambda_2}(H(\lambda_1,\lambda_2))$;

(2) *If* $(\lambda_1,\lambda_2) \dot{<} (\lambda_3,\lambda_4)$, *then* $H(\lambda_1,\lambda_2) \subseteq H(\lambda_3,\lambda_4)$;

(3) $A^{(\alpha_1,\alpha_2)} \supseteq \bigcap_{(\lambda_1,\lambda_2)\dot{<}(\alpha_1,\alpha_2)} H(\lambda_1,\lambda_2), \quad ((\alpha_1,\alpha_2) \neq (0,1))$;

(4) $A^{(\alpha_1,\alpha_2)\cdot} = \bigcup_{(\lambda_1,\lambda_2)\dot{<}(\alpha_1,\alpha_2)} H(\lambda_1,\lambda_2), \quad ((\alpha_1,\alpha_2) \neq (0,1))$.

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