Distances Between Interval-Valued Intuitionistic Fuzzy Sets

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Abstract

In this paper we define the distances between intuitionistic fuzzy sets, interval-valued fuzzy sets and interval-valued intuitionistic fuzzy sets. Four basic distances - Hamming distance, normalized Hamming distance, Euclidean distance and normalised Euclidean distance between these sets are introduced. Also, we present some relations between the distances among different sets.

Keywords: *Fuzzy set, intuitionistic fuzzy set, interval-valued fuzzy set, interval-valued intuitionistic fuzzy sets, Hamming distance, Euclidean distance.*

1 Introduction

In 1986 [1], Atanassov introduced the intuitionistic fuzzy set (IFS) for first time. After three years Atanassov and Gargov [2] introduced the interval-valued intuitionistic fuzzy sets (IVIFS). The distances – Hamming distance, normalised Hamming distance, Euclidean distance, normalised Euclidean distance on intuitionistic fuzzy sets are defined in [4]. Here we propose some new definitions of distances between interval-valued fuzzy sets (IVFS) and IVIFS. In this paper we present some relations between Hamming distance and Euclidean distance among IFS, IVFS, IVIFS.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be an universal set with cardinality n. Let $D[0,1]$ be the set of all closed subintervals of the interval [0,1] and elements of this set are denoted by uppercase letters. If $M \in D[0,1]$ then it can be represented as $M = [M_L, M_U]$, where M_L and M_U are the lower and the upper limits of M. For $M \in D[0,1]$, $\bar{M} = 1 - \bar{M}$ represents the interval $[1 - M_L, 1 - M_U]$ and $W_M = M_U - M_L$ is the width of M.

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A fuzzy set A in $X = \{x\}$ is given by

$$A = \{< x, \mu_A(x) > / x \in X\}$$

where $\mu_A : X \longrightarrow [0,1]$ is the membership function of the fuzzy set A; $\mu_A(x) \in [0,1]$ is the membership of $x \in X$ in A.

An intuitionistic fuzzy set A in $X = \{x\}$ is an expression given by

$$A = \{< x, \mu_A(x), \nu_A(x) > / x \in X\}$$

where $\mu_A : X \longrightarrow [0,1]$, $\nu_A : X \longrightarrow [0,1]$.

With the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. The numbers $\mu_A(x)$, $\nu_A(x) \in [0,1]$ denote respectively the degree of membership and degree of nonmembership of the element x to the set A. The number $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the intuitionistic index of the element x in the set A.

An interval-valued fuzzy set A in X is given by

$$A = \{< x, M_A(x) > / x \in X\}$$

where $M_A : X \longrightarrow D[0,1]$, $M_A(x)$ denote the degree of membership of the element x to the set A.

An interval-valued intuitionistic fuzzy set A in X, is given by

$$A = \{< x, M_A(x), N_A(x) > / x \in X\}$$

where $M_A : X \longrightarrow D[0,1]$, $N_A : X \longrightarrow D[0,1]$.

The intervals $M_A(x)$ and $N_A(x)$ denote the degree of membership and the degree of non-membership of the element x to the set A, where $M_A(x) = [M_{AL}(x), M_{AU}(x)]$ and $N_A(x) = [N_{AL}(x), N_{AU}(x)]$ with the condition $0 \leq M_{AU}(x) + N_{AU}(x) \leq 1$ for all $x \in X$.

2 Distance in IFS, IVFS, IVIFS

In this section we define the four basic distances between IFS, IVFS, IVIFS – the Hamming distance, the normalised Hamming distance, the Euclidean distance and the normalised Euclidean distance. After defining these distances we give some examples.

2.1 Distances on IFS

Let $X = \{x_1, x_2, \ldots, x_n\}$ be the universal set. The Hamming distance between two IFSs A and $B \in X$ is

$$d_{HIFS}(A, B) = \sum_{i=1}^{n} \frac{|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|}{2}$$

and the normalised Hamming distance is

$$d'_{HIFS}(A,B) = \frac{d_{HIFS}(A,B)}{n}.$$

It is clear that $d_{HIFS}(A,B) \leq n$ and $d'_{HIFS}(A,B) \leq 1$. The Euclidean distance between IFSs A, B in X is

$$d_{EIFS}(A,B) = \sqrt{\sum_{i=1}^{n} \frac{(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2}{2}}.$$

The normalised Euclidean distance is

$$d'_{EIFS}(A,B) = \frac{d_{EIFS}(A,B)}{\sqrt{n}}.$$

Obviously, $d_{EIFS}(A,B) \leq \sqrt{n}$ and $d'_{EIFS}(A,B) \leq 1$.

2.2 Distances on IVFS

Let A and B be two IVFSs, then the Hamming distance between A and B is given by

$$d_{HIVFS}(A,B) = \sum_{i=1}^{n} \frac{|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)|}{2}.$$

The normalised Hamming distance between A and B is

$$d'_{HIVFS}(A,B) = \frac{d_{HIVFS}(A,B)}{n}.$$

The Euclidean distance between two IVFSs A and B in X is
$d_{EIVFS}(A,B) = \sqrt{\sum_{i=1}^{n} \frac{(M_{AL}(x_i) - M_{BL}(x_i))^2 + (M_{AU}(x_i) - M_{BU}(x_i))^2}{2}}$.
The normalised Euclidean distance is

$$d'_{EIVFS}(A,B) = \frac{d_{EIVFS}(A,B)}{\sqrt{n}}$$

Again, $d_{HIVFS}(A,B) \leq n$, $d'_{HIVFS}(A,B) \leq 1$ and $d_{EIVFS}(A,B) \leq \sqrt{n}$, $d'_{EIVFS}(A,B) \leq 1$.

2.3 Distances on IVIFS

Let A and B be two IVIFSs in X, $X = \{x_1, x_2, \ldots, x_n\}$ then we define the Hamming distance between A and B as follows:

$$d_{HIVIFS}(A,B) = \sum_{i=1}^{n} \tfrac{1}{4}\{|M_{AL}(x_i) - M_{BL}(x_i)| + |M_{AU}(x_i) - M_{BU}(x_i)| + |N_{AL}(x_i) - N_{BL}(x_i)| + |N_{AU}(x_i) - N_{BU}(x_i)|\}.$$

The normalised Hamming distance is

$$d'_{HIVIFS}(A,B) = \frac{d_{HIVIFS}(A,B)}{n}.$$

The Euclidean distance between two IVIFSs A and B in X is given by
$d_{EIVIFS}(A,B) = \sqrt{\sum_{i=1}^{n} \frac{(M_{AL}(x_i)-M_{BL}(x_i))^2+(M_{AU}(x_i)-M_{BU}(x_i))^2+(N_{AL}(x_i)-N_{BL}(x_i))^2+(N_{AU}(x_i)-N_{BU}(x_i))^2}{4}}$.

The normalised Euclidean distance is

$$d'_{EIVIFS}(A,B) = \frac{d_{EIVIFS}(A,B)}{\sqrt{n}}.$$

Also, on IVIFS, $d_{HIVIFS}(A,B) \leq n$, $d'_{HIVIFS}(A,B) \leq 1$ and $d_{EIVIFS}(A,B) \leq \sqrt{n}$, $d'_{EIVIFS}(A,B) \leq 1$.

Example 1 Let us consider two IFSs A and B in $X = \{a,b,c,d\}$ as follows:
$A = \{< a, 0.7, 0.3 >, < b, 0.4, 0.5 >, < c, 0.2, 0.6 >, < d, 0.9, 0.0 >\}$
$B = \{< a, 0.3, 0.4 >, < b, 0.5, 0.5 >, < c, 0.1, 0.5 >, < d, 0.7, 0.2 >\}$.

The Hamming distance between A and B is
$d_{HIFS}(A,B) = \frac{1}{2}\{|0.7-0.3| + |0.3-0.4| + |0.4-0.5| + |0.5-0.5| + |0.2-0.1| + |0.6-0.9| + |0.9-0.7| + |0-0.2|\} = \frac{1.4}{2} = 0.7$

While the normalised Hamming distance is $d'_{HIFS}(A,B) = 0.7/4 = 0.175$

The Euclidean distance between A and B is
$d_{EIFS}(A,B) = [\{(0.7-0.3)^2 + (0.3-0.4)^2 + (0.4-0.5)^2$
$+(0.5-0.5)^2 + (0.2-0.1)^2 + (0.6-0.9)^2 + (0.9-0.7)^2 + (0-0.2)^2\}/2]^{1/2}$
$= 0.424$

The normalised Euclidean distance is
$d'_{EIFS}(A,B) = \frac{0.424}{4} = 0.106$

Example 2 Let two IVFSs A and B in $X = \{a,b,c,d\}$ as follows:
$A = \{< a, [0.1, 0.6] >, < b, [0, 0.5] >, < c, [0.3, 0.6] >, < d, [0.2, 0.4] >\}$
$B = \{< a, [0.2, 0.7] >, < b, [0.4, 0.6] >, < c, [0, 0.3] >, < d, [0.1, 0.7] >\}$.

The Hamming distance between two IVFSs A and B is $d_{HIVFS}(A,B) = \frac{1}{2}\{|0.1-0.2| + |0.6-0.7| + |0-0.4| + |0.5-0.6| + |0.3-0| + |0.6-0.3| + |0.2-0.1| + |0.4-0.7|\} = \frac{1}{2}(1.7) = 0.85$

The normalised Hamming distance is given by $d'_{HIVFS}(A,B) = \frac{0.85}{4} = 0.2125$

The Euclidean distance between A and B is
$d_{EIVFS}(A,B) = \frac{1}{2}\{(0.1-0.2)^2 + (0.6-0.7)^2 + (0-0.4)^2 + (0.5-0.6)^2 + (0.3-0)^2 + (0.6-0.3)^2 + (0.2-0.1)^2 + (0.4-0.7)^2\}^{1/2} = \sqrt{\frac{1}{2}(0.01 + 0.01 + 0.16 + 0.01 + 0.09 + 0.09 + 0.01 + 0.09)} = \sqrt{0.235} = 0.4847$

The normalised Euclidean distance is $d'_{EIVFS}(A,B) = \frac{0.4847}{4} = 0.121175$

Example 3 Let two IVIFSs A and B in $X = \{a,b,c,d\}$ as follows:
$A = \{< a, [0.2, 0.5], [0, 0.3] >, < b, [0.3, 0.6], [0.2, 0.3] >, < c, [0.1, 0.4], [0.2, 0.3] >,$
$< d, [0, 0.6], [0.1, 0.9] >\}$

$B = \{< a, [0.1, 0.6], [0.3, 0.4] >, < b, [0.2, 0.5], [0.3, 0.4] >, < c, [0.5, 0.6], [0.1, 0.3] >$
$< d, [0.1, 0.5], [0.2, 0.5] >\}$.

Now we calculate the Hamming distance between two IVIFSs A and B as follows:

$d_{HIVIFS}(A, B) = \frac{1}{4}\{|0.2 - 0.1| + |0.5 - 0.6| + |0 - 0.3|$
$+|0.3 - 0.4| + |0.3 - 0.2| + |0.6 - 0.5| + |0.2 - 0.3| + |0.3 - 0.4| + |0.1 - 0.5| + |0.4 - 0.6| + |0.2 -$
$0.1| + |0.3 - 0.3| + |0 - 0.1| + 0.6 - 0.5| + |0.1 - 0.2| + |0.9 - 0.5|\} = \frac{1}{4}(2.4) = 0.6$

The normalised Hamming distance is $d'_{HIVIFS}(A, B) = \frac{d_{HIVIFS}(A,B)}{4} = \frac{0.6}{4} = 0.15$

Let us calculate the Euclidean distance between A and B

$d_{EIVIFS}(A, B) = (\frac{1}{4}\{(0.2 - 0.1)^2 + (0.5 - 0.6)^2 + (0 - 0.3)^2 + (0.3 - 0.4)^2 + (0.3 - 0.2)^2 + (0.6 -$
$0.5)^2 + (0.2 - 0.3)^2 + (0.3 - 0.4)^2 + (0.1 - 0.5)^2$
$+(0.4 - 0.6)^2 + (0.2 - 0.1)^2 + (0.3 - 0.3)^2 + (0 - 0.1)^2 + (0.6 - 0.5)^2 + (0.1 - 0.2)^2$
$+(0.9 - 0.5)^2\})^{\frac{1}{2}} = 0.374$

The normalised Euclidean distance is $d'_{EIVIFS}(A, B) = \frac{0.374}{4} = 0.0935$

3 Some Operations Defined Over the IFS, IVFS and IVIFS

Some operations are defined over the IFS by Atanassov [3]. For every two IFSs A and B in X, $X = \{x_1, x_2, \ldots, x_n\}$ several operations are defined. In this section we introducing only those essential operations which are related to our work.

3.1 On IFS

$A + B = \{< x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x) > /x \in X\}$
$A.B = \{< x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x) > /x \in X\}$
$\Box A = \{< x, \mu_A(x), 1 - \mu_A(x) > /x \in X\}$
$\Diamond A = \{< x, 1 - \nu_A(x), \nu_A(x) > /x \in X\}$
$A \cup B = \{< x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x)) > /x \in X\}$
$A \cap B = \{< x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x)) > /x \in X\}$
$A@B = \{< x, \frac{\mu_A(x)+\mu_B(x)}{2}, \frac{\nu_A(x)+\nu_B(x)}{2} > /x \in X\}$

Let $\alpha, \beta \in [0, 1]$ be two fixed numbers, where $\alpha + \beta \leq 1$. Then the following operations are defined.

$D_\alpha(A) = \{< x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + (1 - \alpha).\pi_A(x) > /x \in X\}$
$F_{\alpha,\beta}(A) = \{< x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x) > /x \in X\}$
$J_{\alpha,\beta}(A) = \{< x, \mu_A(x) + \alpha.\pi_A(x), \beta.\nu_A(x) > /x \in X\}$
$G_{\alpha,\beta}(A) = \{< x, \alpha.\mu_A(x), \beta.\nu_A(x) > /x \in X\}$
$H_{\alpha,\beta}(A) = \{< x, \alpha.\mu_A(x), \nu_A(x) + \beta.\pi_A(x) > /x \in X\}$
$H^*_{\alpha,\beta}(A) = \{< x, \alpha.\mu_A(x), \nu_A(x) + \beta.(1 - \alpha.\mu_A(x) - \nu_A(x)) > /x \in X\}$
$J^*_{\alpha,\beta}(A) = \{< x, \mu_A(x) + \alpha(1 - \mu_A(x) - \beta.\nu_A(x)), \beta.\nu_A(x) > /x \in X\}$

3.2 On IVFS

Let A and B be two IVFSs of X, $X = \{x_1, x_2, \ldots, x_n\}$ then several operations defined over A and B. Here we introducing some of these operations. Let α, β be two fixed number with $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$.

$A+B = \{< x, [M_{AL}(X)+M_{BL}(x) - M_{AL}(x).M_{BL}(x), M_{AU}(x)+M_{BU}(x) - M_{AU}(x).M_{BU}(x)] > /x \in X\}$

$A.B = \{< x, [M_{AL}(x).M_{BL}(x), M_{AU}(x).M_{BU}(x)] > /x \in X\}$

$A@B = \{< x, [\frac{M_{AL}(x)+M_{BL}(x)}{2}, \frac{M_{AU}(x)+M_{BU}(x)}{2}] > /x \in X\}$

$A \cup B = \{< x, [max(M_{AL}(x), M_{BL}(x)), max(M_{AU}(x), M_{BU}(x))] > /x \in X\}$

$A \cap B = \{< x, [min(M_{AL}(x), M_{BL}(x)), min(M_{AU}(x), M_{BU}(x))] > /x \in X\}$

$D_\alpha(A) = \{< x, [M_{AL}(x) + \alpha.W_A(x), M_{AU}(x) + \alpha.W_A(x)] > /x \in X\}$

3.3 On IVIFS

Similarly we shall introduce some new operations on IVIFS [5], taking into account both the degree of membership and the degree of nonmembership of the element $x \in X$ to the set A.

Let A and B be two IVIFSs of X, $X = \{x_1, x_2, \ldots, x_n\}$ then some operations are defined. Here we shall introduce these operations, considering $\alpha, \beta \in [0,1]$ be two fixed numbers and $\alpha + \beta \leq 1$.

We define $\pi_A(x) = 1 - M_{AU}(x) - N_{AU}(x)$, called interval-valued intuitionistic index of the set A. $A+B = \{< x, [M_{AL}(x)+M_{BL}(x) - M_{AL}(x).M_{BL}(x), M_{AU}(x)+M_{BU}(x) - M_{AU}(x)M_{BU}(x)],$
$[N_{AL}(x)N_{BL}(x), N_{AU}(x)N_{BU}(x)] > /x \in X\}$

$A.B = \{< x, [M_{AL}(x)M_{BL}(x), M_{AU}(x)M_{BU}(x)],$
$[N_{AL}(x) + N_{BL}(x) - N_{AL}(x)N_{BL}(x), N_{AU}(x) + N_{BU}(x) - N_{AU}(x)N_{BU}(x)] > /x \in X\}$

$\Box A = \{< x, [M_{AL}(x), M_{AU}(x)], [1 - M_{AL}(x), 1 - M_{AU}(x)] > /x \in X\}$

$\Diamond A = \{< x, [1 - N_{AL}(x), 1 - N_{AU}(x)], [N_{AL}(x), N_{AU}(x)] > /x \in X\}$

$A \cup B = \{< x, [max(M_{AL}(x), M_{BL}(x)), max(M_{AU}(x), M_{BU}(x))],$
$[min(N_{AL}(x), N_{BL}(x)), min(N_{AU}(x), N_{BU}(x))] > /x \in X\}$

$A \cap B = \{< x, [min(M_{AL}(x), M_{BL}(x)), min(M_{AU}(x), M_{BU}(x))],$
$[max(N_{AL}(x), N_{BL}(x)), max(N_{AU}(x), N_{BU}(x))] > /x \in X\}$

$D_\alpha(A) = \{< x, [M_{AL}(x) + \alpha.\pi_A(x), M_{AU}(x) + \alpha.\pi_A(x)], [N_{AL}(x) + (1-\alpha)\pi_A(x),$
$N_{AU}(x) + (1-\alpha)\pi_A(x)] > /x \in X\}$

$F_{\alpha,\beta}(A) = \{< x, [M_{AL}(x) + \alpha.\pi_A(x), M_{AU}(x) + \alpha.\pi_A(x)], [N_{AL}(x) + \beta.\pi_A(x),$
$N_{AU}(x) + \beta.\pi_A(x)] > /x \in X\}$

$J_{\alpha,\beta}(A) = \{< x, [M_{AL}(x) + \alpha.\pi_A(x), M_{AU}(x) + \alpha.\pi_A(x)], [\beta.N_{AL}(x), \beta.N_{AU}(x)] > /x \in X\}$

$G_{\alpha,\beta}(A) = \{< x, [\alpha.M_{AL}(x), \alpha.M_{AU}(x)], [\beta.N_{AL}(x), \beta.N_{AU}(x)] > /x \in X\}$

$H_{\alpha,\beta}(A) = \{< x, [\alpha.M_{AL}(x), \alpha.M_{AU}(x)], [N_{AL}(x) + \beta.\pi_A(x), N_{AU}(x) + \beta.\pi_A(x))] > /x \in X\}$

$H^*_{\alpha,\beta}(A) = \{< x, [\alpha.M_{AL}(x), \alpha.M_{AU}(x)],$
$[N_{AL}(x) + \beta(1 - \alpha.N_{AL}(x) - N_{AL}(x)), N_{AU}(x) + \beta(1 - \alpha.N_{AU}(x) - N_{AU}(x)] > /x \in X\}$
$J^*_{\alpha,\beta}(A) = \{< x, [M_{AL}(x) + \alpha.(1 - M_{AL}(x) - \beta.N_{AL}(x)),$
$M_{AU}(x) + \alpha(1 - M_{AU}(x) - \beta.N_{AU}(x))], [\beta.N_{AL}(x), \beta.N_{AU}(x)] > /x \in X\}$
$A@B = \{< x, [\frac{M_{AL}(x)+M_{BL}(x)}{2}, \frac{M_{AU}(x)+M_{BU}(x)}{2}], [\frac{N_{AL}(x)+N_{BL}(x)}{2}, \frac{N_{AU}(x)+N_{BU}(x)}{2}] > /x \in X\}$

4 Some Relations Between Distances

In this section we introduced some relations between Hamming distance and Euclidean distance on IFS, IVFS, IVIFS.

Theorem 1 *For any two IFSs A and B,*
(i) $d_{HIFS}(A, \Box A) = d_{HIFS}(A, \Diamond A)$
(ii) $d_{HIFS}(A, A.B) = d_{HIFS}(B, A + B)$
(iii) $d_{HIFS}(A, A^n) = d_{HIFS}(A, nA)$
(iv) $d_{HIFS}(\Box A, \Diamond A) = 2d_{HIFS}(A, \Box A) = 2d_{HIFS}(A, \Diamond A)$
(v) $d_{HIFS}(A + B, A@B) = d_{HIFS}(A.B, A@B)$
(vi) $d_{HIFS}(G_{\alpha,\beta}(A), F_{\alpha,\beta}(A)) = d_{HIFS}(H_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$
(vii) $d_{HIFS}(F_{\alpha,\beta}(A), H_{\alpha,\beta}(A)) = d_{HIFS}(G_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$

Proof: (i) We have $d_{HIFS}(A, \Box A) = \frac{1}{2}\sum_{i=1}^{n} |\nu_A(x_i) - 1 + \mu_A(x_i)| = \frac{1}{2}\sum_{i=1}^{n} |1 - \mu_A(x_i) - \nu_A(x_i)|$ and
$d_{HIFS}(A, \Diamond A) = \frac{1}{2}\sum_{i=1}^{n}(|\mu_A(x_i) - 1 + \nu_A(x_i)| + |\nu_A(x_i) - \nu_A(x_i)| = \frac{1}{2}\sum_{i=1}^{n} |1 - \mu_A(x_i) - \nu_A(x_i)|$
Therefore, $d_{HIFS}(A, \Box A) = d_{HIFS}(A, \Diamond A)$.
(ii) $d_{HIFS}(A, A.B) = \frac{1}{2}\sum_{i=1}^{n}(|\mu_A(x_i) - \nu_A(x_i)\nu_B(x_i)| + |\nu_A(x_i) - \nu_A(x_i) - \nu_B(x_i) + \nu_A(x_i)\nu_B(x_i)|)$
$= \frac{1}{2}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_A(x_i)\mu_B(x_i)| + |\nu_B(x_i) - \nu_A(x_i)\nu_B(x_i)|)$
$d_{HIFS}(B, A+B) = \frac{1}{2}\sum_{i=1}^{n}(|\mu_B(x_i) - \mu_A(x_i) - \mu_B(x_i) + \mu_A(x_i)\mu_B(x_i)| + |\nu_B(x_i) - \nu_A(x_i)\nu_B(x_i)|) =$
$\frac{1}{2}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_A(x_i)\mu_B(x_i)| + |\nu_B(x_i) - \nu_A(x_i)\nu_B(x_i)|)$.
Therefore, $d_{HIFS}(A, A.B) = d_{HIFS}(B, A + B)$.
(iii) When $A = B$ we have from (ii),
$d_{HIFS}(A, A.A) = d_{HIFS}(A, A + A)$
or, $d_{HIFS}(A, A^2) = d_{HIFS}(A, 2A)$
In general, for any positive integer n, $d_{HIFS}(A, A^n) = d_{HIFS}(A, nA)$
(iv) Now, $d_{HIFS}(\Box A, \Diamond A) = \frac{1}{2}\sum_{i=1}^{n}(|\mu_A(x_i) - 1 + \nu_A(x_i)| + |1 - \mu_A(x_i) - \nu_A(x_i)|)$
$= \sum_{i=1}^{n} |1 - \mu_A(x_i) - \nu_A(x_i)|$
From relation (i), $d_{HIFS}(A, \Box A) = d_{HIFS}(A, \Diamond A)$
Therefore, $d_{HIFS}(\Box A, \Diamond A) = 2d_{HIFS}(A, \Box A) = 2d_{HIFS}(A, \Diamond A)$
(v) $d_{HIFS}(A + B, A@B) = \frac{1}{2}\sum_{i=1}^{n}(|\mu_A(x_i) + \mu_B(x_i) - \mu_A(x_i)\mu_B(x_i) - \frac{\mu_A(x_i)+\mu_B(x_i)}{2}|$
$+ |\nu_A(x_i)\nu_B(x_i) - \frac{\nu_A(x_i)+\nu_B(x_i)}{2}|)$

$=\frac{1}{2}\sum_{i=1}^{n}(|\frac{\mu_A(x_i)+\mu_B(x_i)}{2}-\mu_A(x_i)\mu_B(x_i)|+|\frac{\nu_A(x_i)+\nu_B(x_i)}{2}-\nu_A(x_i)\nu_B(x_i)|)$
$d_{HIFS}(A.B, A@B)=\frac{1}{2}\sum_{i=1}^{n}(|\mu_A(x_i)\mu_B(x_i)-\frac{\mu_A(x_i)+\mu_B(x_i)}{2}|$
$+|\nu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\nu_B(x_i)-\frac{\nu_A(x_i)+\nu_B(x_i)}{2}|)$
$=\frac{1}{2}\sum_{i=1}^{n}(|\frac{\mu_A(x_i)+\mu_B(x_i)}{2}-\mu_A(x_i)\mu_B(x_i)|+|\frac{\nu_A(x_i)+\nu_B(x_i)}{2}-\nu_A(x_i)\nu_B(x_i)|)$
Therefore, $d_{HIFS}(A+B, A@B)=d_{HIFS}(A.B, A@B)$.
(vi) $d_{HIFS}(G_{\alpha,\beta}(A), F_{\alpha,\beta}(A))=\frac{1}{2}\sum_{i=1}^{n}(|\alpha\mu_A(x_i)-\mu_A(x_i)-\alpha\pi_A(x_i)|+|\beta\nu_A(x_i)-\nu_A(x_i)-\beta\pi_A(x_i)|)$
$=\frac{1}{2}\sum_{i=1}^{n}(|\alpha(\mu_A(x_i)-\pi_A(x_i))-\mu_A(x_i)|+|\beta(\nu_A(x_i)-\pi_A(x_i))-\nu_A(x_i)|)$
$d_{HIFS}(H_{\alpha,\beta}(A), J_{\alpha,\beta}(A))=\frac{1}{2}\sum_{i=1}^{n}(|\alpha\mu_A(x_i)-\mu_A(x_i)-\alpha\pi_A(x_i)|+|\nu_A(x_i)+\beta\pi_A(x_i)-\beta\nu_A(x_i)|)$
$=\frac{1}{2}\sum_{i=1}^{n}(|\alpha(\mu_A(x_i)-\pi_A(x_i))-\mu_A(x_i)|+|\beta(\nu_A(x_i)-\pi_A(x_i))-\nu_A(x_i)|)$
Hence, $d_{HIFS}(G_{\alpha,\beta}(A), F_{\alpha,\beta}(A))=d_{HIFS}(H_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$
(vii) $d_{HIFS}(F_{\alpha,\beta}(A), H_{\alpha,\beta}(A))=\frac{1}{2}\sum_{i=1}^{n}(|\mu_A(x_i)+\alpha\pi_A(x_i)-\alpha\mu_A(x_i)|+|\nu_A(x_i)+\beta\pi_A(x_i)-\nu_A(x_i)-\beta\pi_A(x_i)|)$
$=\frac{1}{2}\sum_{i=1}^{n}|\alpha(\mu_A(x_i)-\pi_A(x_i))-\mu_A(x_i)|$
$d_{HIFS}(G_{\alpha,\beta}(A), J_{\alpha,\beta}(A)=\frac{1}{2}\sum_{i=1}^{n}(|\alpha\mu_A(x_i)-\mu_A(x_i)\alpha\pi_A(x_i)|+|\beta\nu_A(x_i)-\beta\nu_A(x_i)|)$
$=\frac{1}{2}\sum_{i=1}^{n}(|\alpha(\mu_A(x_i)-\pi_A(x_i))-\mu_A(x_i)|$.

Therefore, $d_{HIFS}(F_{\alpha,\beta}(A), H_{\alpha,\beta}(A))=d_{HIFS}(G_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$.

Theorem 2 *For any two IFSs A and B,*
(i) $d_{EIFS}(A, \Box A)=d_{EIFS}(A, \Diamond A)$
(ii) $d_{EIFS}(A, A.B)=d_{EIFS}(B, A+B)$
(iii) $d_{EIFS}(A, A^n)=d_{EIFS}(A, nA)$
(iv) $d_{EIFS}(\Box A, \Diamond A)=\sqrt{2}d_{EIFS}(A, \Box A)=\sqrt{2}d_{EIFS}(A, \Diamond A)$
(v) $d_{EIFS}(A+B, A@B)=d_{EIFS}(A.B, A@B)$
(vi) $d_{EIFS}(G_{\alpha,\beta}(A), F_{\alpha,\beta}(A))=d_{EIFS}(H_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$
(vii) $d_{EIFS}(F_{\alpha,\beta}(A), H_{\alpha,\beta}(A))=d_{EIFS}(G_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$

Proof: Same as Theorem 1.

Theorem 3 *For any two IVFSs A and B,*
(i) $d_{HIVFS}(A, A.B)=d_{HIVFS}(B, A+B)$
(ii) $d_{HIVFS}(A, A^n)=d_{HIVFS}(A, nA)$ *for any integer n.*
(iii) $d_{HIVFS}(A+B, A@B)=d_{HIVFS}(A.B, A@B)$

Proof: Same as Theorem 1.

Theorem 4 *For any two IVFSs A and B,*
(i) $d_{EIVFS}(A, A.B)=d_{EIVFS}(B, A+B)$
(ii) $d_{EIVFS}(A, A^n)=d_{EIVFS}(A, nA)$
(iii) $d_{EIVFS}(A+B, A@B)=d_{EIVFS}(A.B, A@B)$

Proof: Same as Theorem 1.

Theorem 5 *For any two IVIFSs A and B,*
(i) $d_{HIVIFS}(A, \Box A) = d_{HIVIFS}(A, \Diamond A)$
(ii) $d_{HIVIFS}(A, A.B) = d_{HIVIFS}(B, A + B)$
(iii) $d_{HIVIFS}(A, A^n) = d_{HIVIFS}(A, nA)$ *for any integer n.*
(iv) $d_{HIVIFS}(\Box A, \Diamond A) = 2d_{HIVIFS}(A, \Box A,) = 2d_{HIVIFS}(A, \Diamond A)$
(v) $d_{HIVIFS}(A + B, A@B) = d_{HIVIFS}(A.B, A@B)$
(vi) $d_{HIVIFS}(G_{\alpha,\beta}(A), F_{\alpha,\beta}(A)) = d_{HIVIFS}(H_{\alpha,\beta}(A), J_{\alpha,\beta}(A)$
(vii) $d_{HIVIFS}(F_{\alpha,\beta}(A), H_{\alpha,\beta}(A)) = d_{HIVIFS}(G_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$

Proof: (ii) $d_{HIVIFS}(A, A.B)$
$= \sum_{i=1}^{n} \frac{1}{4}\{|M_{AL}(x_i) - M_{AL}(x_i).M_{BL}(x_i)| + |M_{AU}(x_i) - M_{AU}(x_i).M_{BU}(x_i)|$
$+|N_{AL}(x_i) - N_{AL}(x_i) - N_{BL}(x_i) + N_{AL}(x_i).N_{BL}(x_i)| + |N_{AU}(x_i) - N_{AU}(x_i) - N_{BU}(x_i)$
$+N_{AU}(x_i).N_{BU}(x_i)|\}$
$= \sum_{i=1}^{n} \frac{1}{4}\{|M_{AL}(x_i) - M_{AL}(x_i).M_{BL}(x_i)| + |M_{AU}(x_i) - M_{AU}(x_i).M_{BU}(x_i)|$
$+|N_{BL}(x_i) - N_{AL}(x_i).N_{BL}(x_i)| + |N_{BU}(x_i) - N_{AU}(x_i).N_{BU}(x_i)|\}$
$d_{HIVIFS}(B, A + B)$
$= \sum_{i=1}^{n} \frac{1}{4}\{|M_{BL}(x_i) - M_{AL}(x_i) - M_{BL}(x_i) + M_{AL}(x_i).M_{BL}(x_i)| + |M_{BU}(x_i) - M_{AU}(x_i) - M_{BU}(x_i) +$
$M_{AU}(x_i).M_{BU}(x_i)| + |N_{BL}(x_i) - N_{AL}(x_i).N_{BL}(x_i)| + |N_{BU}(x_i) - N_{AU}(x_i).N_{BU}(x_i)|\}$
$= \sum_{i=1}^{n} \frac{1}{4}\{|M_{AL}(x_i) - M_{AL}(x_i).M_{BL}(x_i)| + |M_{AU}(x_i) - M_{AU}(x_i).M_{BU}(x_i)|$
$+|N_{BL}(x_i) - N_{AL}(x_i).N_{BL}(x_i)| + |N_{BU}(x_i) - N_{AU}(x_i).N_{BU}(x_i)|\}$
Therefore, $d_{HIVIFS}(A, A.B) = d_{HIVIFS}(B, A + B)$.
(v) $d_{HIVIFS}(A + B, A@B)$
$= \sum_{i=1}^{n} \frac{1}{4}\{|M_{AL}(x_i) + M_{BL}(x_i) - M_{AL}(x_i).M_{BL}(x_i) - \frac{M_{AL}(x_i)+M_{BL}(x_i)}{2}| + |M_{AU}(x_i) + M_{BU}(x_i) -$
$M_{AU}(x_i).M_{BU}(x_i) - \frac{M_{AU}(x_i)+M_{BU}(x_i)}{2}| + |N_{AL}(x_i).N_{BL}(x_i) - \frac{N_{AL}(x_i)+N_{BL}(x_i)}{2}| + |N_{AU}(x_i).N_{BU}(x_i) -$
$\frac{N_{AU}(x_i)+N_{BU}(x_i)}{2}|\})$
$= \sum_{i=1}^{n}(\frac{1}{4}\{|\frac{M_{AL}(x_i)+M_{BL}(x_i)}{2} - M_{AL}(x_i).M_{BL}(x_i)| + |\frac{M_{AU}(x_i)+M_{BU}(x_i)}{2} - M_{AU}(x_i).M_{BU}(x_i)|$
$+|\frac{N_{AL}(x_i)+N_{BL}(x_i)}{2} - N_{AL}(x_i).N_{BL}(x_i)|$
$+|\frac{N_{AU}(x_i)+N_{BU}(x_i)}{2} - N_{AU}(x_i).N_{BU}(x_i)|\}$
$d_{HIVIFS}(A.B, A@B)$
$= \sum_{i=1}^{n} \frac{1}{4}\{|M_{AL}(x_i).M_{BL}(x_i) - \frac{M_{AL}(x_i)+M_{BL}(x_i)}{2}| + |M_{AU}(x_i).M_{BU}(x_i) - \frac{M_{AU}(x_i)+M_{BU}(x_i)}{2}|$
$+|N_{AL}(x_i) + N_{BL}(x_i) - N_{AL}(x_i).N_{BL}(x_i) - \frac{N_{AL}(x_i)+N_{BL}(x_i)}{2}|$
$+|N_{AU}(x_i) + N_{BU}(x_i) - N_{AU}(x_i).N_{BU}(x_i) - \frac{N_{AU}(x_i)+N_{BU}(x_i)}{2}|\}$
$= \sum_{i=1}^{n} \frac{1}{4}\{\frac{M_{AL}(x_i)+M_{BL}(x_i)}{2} - M_{AL}(x_i).M_{BL}(x_i)| + |\frac{M_{AU}(x_i)+M_{BU}(x_i)}{2} - M_{AU}(x_i).M_{BU}(x_i)|$
$+|\frac{N_{AL}(x_i)+N_{BL}(x_i)}{2} - N_{AL}(x_i).N_{BL}(x_i)| + |\frac{N_{AU}(x_i)+N_{BU}(x_i)}{2} - N_{AU}(x_i)N_{BU}(x_i)|\}$
Therefore, $d_{HIVIFS}(A + B, A@B) = d_{HIVIFS}(A.B, A@B)$

Theorem 6 *For any two IVIFSs Aand B,*
(i) $d_{EIVIFS}(A, \Box A) = d_{EIVIFS}(A, \Diamond A)$

(ii) $d_{EIVIFS}(A, A.B) = d_{EIVIFS}(B, A + B)$
(iii) $d_{EIVIFS}(A, A^n) = d_{EIVIFS}(A, nA)$ *for any integer* n.
(iv) $d_{EIVIFS}(\Box A, \Diamond A) = \sqrt{2} d_{EIVIFS}(A, \Box A) = \sqrt{2} d_{EIVIFS}(A, \Diamond A)$
(v) $d_{EIVIF}(A + B, A@B) = d_{EIVIFS}(A.B, A@B)$
(vi) $d_{EIVIFS}(G_{\alpha,\beta}(A), F_{\alpha,\beta}(A)) = d_{EIVIFS}(H_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$
(vii) $d_{EIVIFS}(F_{\alpha,\beta}(A), H_{\alpha,\beta}(A)) = d_{EIVIFS}(G_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$

Proof: (vi) $d_{EIVIFS}(G_{\alpha,\beta}(A), F_{\alpha,\beta}(A))$
$= \frac{1}{4}\sum_{i=1}^{n}\{(\alpha.M_{AL}(x_i) - M_{AL}(x_i) - \alpha.\pi_A(x_i))^2 + (\alpha.M_{AU}(x_i) - M_{AU}(x_i) - \alpha.\pi_A(x_i)$
$+(\beta.N_{AL}(x_i) - N_{AL}(x_i) - \beta.\pi_A(x_i))^2 + (\beta.N_{UA}(x_i) - N_{AU}(x_i) - \beta.\pi_A(x_i))^2\}^{\frac{1}{2}}$
$= \frac{1}{4}\sum_{i=1}^{n}[\{\alpha(M_{AL}(x_i) - \pi_A(x_i)) - M_{AL}(x_i)\}^2 + \{\alpha(M_{AU}(x_i) - \pi_A(x_i)) - M_{AU}(x_i)\}^2$
$+\{\beta(N_{AL}(x_i) - \pi_A(x_i)) - N_{AL}(x_i)\}^2 + \{\beta(N_{AU}(x_i) - \pi_A(x_i)) - N_{AU}(x_i)\}^2]^{\frac{1}{2}}$.
$d_{EIVIFS}(H_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$
$= \frac{1}{4}\sum_{i=1}^{n}\{(\alpha.M_{AL}(x_i) - M_{AL}(x_i) - \alpha.\pi_A(x_i))^2 + (\alpha.M_{AU}(x_i) - M_{AU}(x_i) - \alpha.\pi_A(x_i))^2 + (N_{AL}(x_i) +$
$\beta.\pi_A(x_i) - \beta.N_{AL}(x_i))^2 + (N_{AU}(x_i) + \beta.\pi_A(x_i) - \beta.N_{AU}(x_i))^2\}^{\frac{1}{2}}$
$= \frac{1}{4}\sum_{i=1}^{n}[\{\alpha(M_{AL}(x_i) - \pi_A(x_i)) - M_{AL}(x_i)\}^2 + \{\alpha(M_{AU}(x_i) - \pi_A(x_i)) - M_{AU}(x_i)\}^2$
$+\{\beta(N_{AL}(x_i) - \pi_A(x_i)) - N_{AL}(x_i)\}^2 + \{\beta(N_{AU}(x_i) - \pi_A(x_i)) - N_{AU}(x_i)\}^2]^{\frac{1}{2}}$
Therefore, $d_{EIVIFS}(G_{\alpha,\beta}(A), F_{\alpha,\beta}(A)) = d_{EIVIFS}(H_{\alpha,\beta}(A), J_{\alpha,\beta}(A))$

Other proofs are similar.

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