Some Results on Lukasiewicz generalized operators

Wang Hong Wei

(Department of Mathematics , Lin Yi teachers College . Lin Yi 276005 , China)

Abstract : In this paper ,some results of Lukasiewicz generalized operators are obtained. We provide a list of properties for λ and μ functions.

Keywords : Implication operators , strong negation , residuated implication .

In the fuzzy logic literature, that extends classical Boolean implication,the classical ones are considered $p \rightarrow q$ for proposition p and q, are defined (Table 1)

Table 1

$\rightarrow$	0	1
0	1	1
1	0	1

Thus, the maps considered are $I : [0,1] \times [0,1] \rightarrow [0,1]$, in such a way that the true valued of $(p \rightarrow q)$ are given By $I(v(p), v(q))$. Dbois and Prade proposed an interesting classification operators used in the bibliography.

From an axiomatics point of view, Dubois and Prade provide a list of properties which these operators must satisfy [1-3]. Concerning the implication operators, Lukasiewicz is the unique operator that verifies the whole of Duboi and Prade's properties.[2-3]

In[4],P.Burillo et al. introduced Lukasiewicz generalized is the unique operators as maps given by $L(a,b)= \min(1, \lambda(a)+ \mu(b))$, $(a,b) \in [0,1] \times [0,1]$, with λ and μ functions from [0,1] to [0,1] . In this paper ,some properties on Lukasiewicz generalized operators are obtained.

Definition 1[4] : Lukasiewicz generalized operators are the maps $L: [0,1] \times [0,1] \rightarrow [0,1]$.
such that $L(a,b)=\min(1, \lambda(a)+ \mu(b))$, $\forall$ a,b $\in [0,1]$, where functions $\lambda , \mu :[0,1] \rightarrow [0,1]$

verifing $\lambda(0)+\mu(1)=1$ and $\lambda(1)+\mu(0)=0$.

The limit conditions required for the functions λ and μ are sufficient to assure the operator L is a generalization of classical implication.

Theorem 1[4]. Under the conditions stated in Definition 1, the following properties are verified:

(1) $L(0,b)=1 \quad \forall b\in[0,1]$

(2) $L(a,0)=\lambda(a) \quad \forall a\in[0,1]$

(3) $L(a,1)=1 \quad \forall a\in[0,1]$

(4) $L(1,b)=\mu(b) \quad \forall b\in[0,1]$

(5) $L(a,b)=1 \Leftrightarrow \lambda(a)+\mu(b)\geq 1 \quad \forall a,b\in[0,1]$

(6) $L(a,b)=0 \Leftrightarrow \lambda(a)=1$ and $\mu(b)=0$

Remark 1 : The valued L(a,b) depends on values a and b.

Theorem 2. Let L be defined with the conditions of Definition 1. Then L verifies

$L(a,b)=L(\sqrt{1-b},1-a^2), \quad \forall a,b\in[0,1]$, iff $\lambda(p)=\mu(1-p^2)$.

Proof. ($\Rightarrow$) If $L(a,b)=L(\sqrt{1-b},1-a^2), \quad \forall a,b\in[0,1]$,

then we have that $L(a,0)=L(1,1-a^2), \quad \forall a\in[0,1]$ (b=0)

and ,by Theorem 1, that $L(a,0)=\lambda(a)$, and $L(1,1-a^2)=\mu(1-a^2) \quad \forall a\in[0,1]$

therefore $\lambda(a)=\mu(1-a^2)$.

($\Leftarrow$) Conversely, if $\lambda(p)=\mu(1-p^2)$, $\mu(p)=\lambda(\sqrt{1-p})$, $\forall p\in[0,1]$,

then $L(a,b)=\min(1, \lambda(a)+\mu(b))$

$=\min(1, \mu(1-a^2)+\lambda(\sqrt{1-b}))$

$=\min(1, \lambda(\sqrt{1-b})+\mu(1-a^2))=L(\sqrt{1-b}, 1-a^2)$.

Remark 2:

① $L(0,0)=L(1,1)=L(0,1)=1$ (Theorem 1 (1))

② $\lambda(0)=\mu(1)=1, \lambda(1)=\mu(0)=0$.

Theorem 3. $L(a,b)=1 \Leftrightarrow a\leq b$, then , "$\leq$" is a partial order.

Proof. Since $\min(1, \lambda(a)+\mu(b))=1 \Leftrightarrow \lambda(a)+\mu(b)\geq 1$,

$\Leftrightarrow \mu(1-a^2)+\mu(b)\geq 1$, so $a \le b \Leftrightarrow \mu(1-a^{2)}+ \mu(b) \ge 1$ „

if $a \le b \Leftrightarrow \mu(1-a^2)+ \mu(b) \ge 1$,

if $b \le c \Leftrightarrow \mu(1-b^2)+ \mu(c) \ge 1$,

since $b \le b$ $\mu(1-b^2)+ \mu(b) \ge 1$, so $\mu(1-a^2)+ \mu(c) + \mu(1-b^2)+ \mu(b) \ge 2$,

hence $\mu(1-a^2)+ \mu(c) \ge 1$, so $a \le c$.

Theorem 4. $\lambda, \mu :[0,1] \to [0,1]$, if $\lambda(a)=1-a^2$, then $\lambda(1-a^2)= a^2$.

Proof. If $\lambda(a)=1-a^2$, $\forall a \in [0,1]$,

$\Rightarrow \lambda(1-a^2) = 1-(1-a^2)^2 = 2a^2-a^4 = a^2(2-a^2)$, and for any $a \in [0,1]$,

$2-a^2 \ge 1$, then we have: $\lambda(1-a^2) \ge a^2$.

Theorem 5. Let $\lambda, \mu :[0,1] \to [0,1]$, if $\lambda(a)=1-a^2$ and $a \le b$, then $\mu(b) \ge a^2$.

Proof. In fact, if $a \le b$, $\forall a,b \in [0,1]$, by Theorem 3,

that $a \le b$ iff $\lambda(a)+\mu(b) \ge 1$, and by condition, $\lambda(a)=1-a^2$,

therefore $1-a^2+ \mu(b) \ge 1$, result $\mu(b) \ge a^2$.

Let $\lambda :[0,1] \to [0,1]$ satisfies following Table 2:

Table 2

1.	$\lambda(0)=1$ and $\lambda(1)=0$
2.	λ nonincreasing and μ nondecreasing
3.	$\lambda(p)= \mu(1-p)$ $p \in [0,1]$
4.	$p \le q \Leftrightarrow \lambda(p) + \lambda(1-q) \ge 1$
5.	λ continuous

We will give now a theorem of characterization of the function λ that satisfies conditions of Table 2.

Theorem 6. Let L satisfy the conditions of Definition 1and λ verifies Table 2, then the following property holds for any $a,b,c \in [0,1]$:

if $\lambda(a)=1-a$, then $L(a,L(b,c))= L(b,L(a,c))$.

Proof. In fact, if $\lambda(a)=1-a$, $\forall a \in [0,1]$,

then $L(a,L(b,c))=\min(1, \lambda(a)+ \lambda(1-\min(1, \lambda(b)+ \lambda(1-c)))$

$= \min (1, 1\text{-}a+1\text{-}(1\text{-}\min(1,1\text{-}b+c)))$

$= \min (1, 1\text{-}a+\min(1,1\text{-}b+c)))$

$L(b,L(a,c))= \min (1, 1\text{-}b+\min(1,1\text{-}a+c)))$

There are three possible different cases :

① $a, b \leq c \Rightarrow L(a,L(b,c))=1= L(b,L(a,c))$

② $a, b \geq c \Rightarrow L(a,L(b,c))= \min (1, 1\text{-}a+1\text{-}b+c)=L(b,L(a,c))$

③ $a\leq c \leq b$, then , $L(a,L(b,c))=1= L(b,L(a,c))$.

Therefore , when $\lambda (a)=1\text{-}a$, then $L(a,L(b,c)) = L(b,L(a,c))$, $\lambda (x)$ is a strong negation , $\lambda (x)=1\text{-}x$.

Now , if each $\lambda :[0,1] \to [0,1]$, and for each summand $([a,b],L)$, we define a strong negation λ_a^b as follows:

$$\lambda_a^b (x)= \frac{\lambda(x(b-a)+a)-a}{b-a} .$$

Straightforward calculation shows that λ_a^b is indeed a strong negation .

In fact ,

$$\lambda_a^b (0)=\frac{\lambda(a)-a}{b-a} = \frac{b-a}{b-a} =1,$$

$$\lambda_a^b (1)=\frac{\lambda(b)-a}{b-a} = \frac{a-a}{b-a} ==0.$$

Theorem 7. Let $T=<\{[a,b],L\}>$, where $a= \lambda (b)$,

then $I_L(u,0)= \lambda_a^b$ holds for all $u \in [0,1]$ if and only if $I_T(x,a)= \lambda (x)$ holds for all $x\in [a,b]$.

Proof . Straightforward verification by using the definition of the residuated implication, $I_T(x,y):=\text{Sup}\{z\in [0,1] : T(x,z)\leq y\}$, and the definition of λ_a^b and the increasing linear bijection between [0,1]and [a,b] as follows :

$I_L(u,0)= \lambda_a^b (u)$ holds for all $u \in [0,1]$

if and only if

$$\frac{\lambda(x(b-a)+a)-a}{b-a}$$

$\text{Sup}\{ z\in[0,1] \mid T(x,z) \le 0\}=$ holds for all $u \in[0,1]$.

That is ,if and only if (with $x:=u(b-a)+a$, and $t:=x(b-a)+a$)

$$\text{Sup}\{ \frac{t-a}{b-a} \mid t \in[a,b] ,T(\frac{x-a}{b-a}, \frac{t-a}{b-a}) \le 0\}= \frac{\lambda(x)-a}{b-a} \quad \text{holds for all } x\in[a,b].$$

This is equivalent to

$$\text{Sup}\{t \mid t\in[a,b] , T(x,t) \le a\}= \lambda(x) \quad \text{holds for all } x\in[a,b] ,$$

which is equivalent to

$$I_T(x,a)= \lambda(x) \quad \text{for all } x\in[a,b].$$

References :

[1] D.Dubois , H.Prade , Fuzzy Sets and Systems :Theory and Applications , Mathematics in science and Engineering , Academic Press ,New York,1980.

[2] D.Dubois , H.Prade , Fuzzy Sets in approximate reasoning , Part 1:inference with possibility distributions , Fuzzy sets and Systems:40(1991)143-202.

[3] D.Dubois , L.Jérō me , H.Prade , Fuzzy sets in approximate reasoning ,Part 2:logical approaches , Fuzzy Sets and Systems 40(1991)203-244.

[4] P.Burillo , N.Frago , R.Fuentes , Inclusion grade and fuzzy implication operators , Fuzzy Sets and Systems,114(2000)41 7-429.