SOME RESULTS ON FIXED POINTS

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Abstract: Recently, Kada et[1]introduced the concept of w — distance on a metric spact. In this paper, we prove fixed point theorems in complete metric space introduced the concept of a w — distance. These theorems extended Fisher's fixed point theorem[2], Jeong — sheok ume fixed point theorem[3].

KEY WORDS: Fixed point, complete metric spaces, distance

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1. PRELIMINARIES

Throughout this paper we denote by N the set of all positive integers.

DEFINITION1. 1. [1] Let X be a metric space with d. Then a function $p: X \times X \to [0,\infty]$ is called a w — distance on X if the following are satisfied:

(1) $p(x,z) \leqslant p(x,y) + p(y,z)$ for any $x,y,z \in X$;

(2) for any $x \in X, p(x,\cdot): X \to [0,\infty)$ is lower semicontinuous;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leqslant \delta$ and $p(z,y) \leqslant \delta$ imply $d(x,y) \leqslant \varepsilon$.

LEMMA1. 2[1] Let X be metric space with d and let p be a w — distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, Let $\{\alpha_n\}$and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to o, and let $x,y,z \in X$.

Then the following hold:

(i) If $p(x_n,y) \leqslant \alpha_n$ and $p(x_n,z) \leqslant \beta_n$ for any $n \in N$, the $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;

(ii) If $p(x_n,y_n) \leqslant \alpha_n$ and $p(x_n,z) \leqslant \beta_n$ for any $n \in N$, then $\{y_n\}$ converges to z;

(iii) If $p(x_n,x_m) \leqslant \alpha_n$, for any $n,m \in N$ with $m > n$, then $\{x_n\}$is a cauchy sequence;

(iv) If $p(y,x_n) \leqslant \alpha_n$ for any $n \in N$, then $\{x_n\}$ is a cauchy sequence.

2. MAIN RESULTS

Theorem 2. 1. Let (X,d) and (Y,ρ) be complete metric space. Let p_1 be a w — distance on X, p_2 be a w — distance on Y. If T is a continuous mapping of X into Y and S is a mapping of Y into X Satisfying the inequalities

$$p_1(STx,STx') \leqslant C\max\{p_1(x,x'),p_1(x,STx),p_1(x',STx'),p_2(Tx,Tx')\}$$

$$p_2(TSy,TSy') \leqslant C\max\{p_2(y,y'),p_2(y,TSy),p_2(y',TSy'),p_1(Sy,Sy')\} \qquad (1.1)$$

for all x,x' in X and y,y' in Y, where $0<C<1$, then ST has a unique fixed Point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

PROOF. Let x be an arbitrary in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by

$$(ST)^n x = x_n,\ T(ST)^{n-1}x = y_n$$

for $n = 1,2,\cdots$. Then

$$p_1(x_n,x_{n+1}) = p_1(STx_{n-1},STx_n) \leqslant C\max\{p_1(x_{n-1},x_n),p_1(x_{n-1},x_n),p_1(x_n,x_{n+1}),p_2(y_n,y_{n+1})\}$$

$$= C\max\{p_1(x_{n-1},x_n),p_2(y_n,y_{n+1})\} \qquad (1.2)$$

and similarly

$$p_2(y_n,y_{n+1}) \leqslant C\max\{p_2(y_{n-1},y_n),p_1(x_{n-1},x_n)\} \qquad (1.3)$$

It now follows easily by induction that

$$p_1(x_n,x_{n+1}) \leqslant C^n\max\{p_1(x,x_1),p_2(y_1,y_2)\} \qquad (1.4)$$

$$p_2(y_n,y_{n+1}) \leqslant C^{n-1}\max\{p_1(x,x_1),p_2(y_1,y_2)\}$$

for $n = 1,2,\cdots$. If $n < m$, then by (1. 2)

$$p_1(x_n,x_m) \leqslant p_1(x_n,x_{n+1}) + p_1(x_{n+1},x_{n+2}) + \cdots\cdots P_1(x_{m-1},x_m)$$

$$\leqslant (C^n + C^{n+1} + \cdots\cdots + C^{m-1})\max\{p_1(x,x_1),p_2(y_1,y_2)\}$$

$$= \frac{C^n}{1-C}\max\{p_1(x,x_1),p_2(y_1,y_2)\} \to 0(n\to\infty)$$

Since $0 < C < 1$, from lemma 1. 2 $\{x_n\}$ is Cauchy sequence, with a limit z in X. Similarly, $\{y_n\}$ is cauchy sequence with a limit w in Y.

We now have on using the continuity of T

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_{n-1} = Tz = w$$

Further, appling inequality (1. 1) we get

$$p_1(STz,x_n) = p_1(STz,STx_{n-1})$$

$$\leqslant C\max\{p_1(z,x_{n-1}),p_1(z,STz),p_1(x_{n-1},x_n),p_2(Tz,y_n)\}$$

and on letting n tend to infinity we have

$$p_1(STz,z) \leqslant C\ \mathrm{mxa}\{p_1(z,z),p_1(z,STz),p_1(z,z),p_2(Tz,w)\} \qquad (1.5)$$

Letting n tend to infinity in inequality (1.2) and (1.3), it follows that

$$p_1(z,z) \leqslant C \max\{p_1(z,z), p_2(w,w)\}$$

$$p_2(w,w) \leqslant C \max\{p_2(w,w), p_1(z,z)\}$$

from which it follows that

$$p_1(z,z) = 0,\ p_2(w,w) = 0.$$

From inequality (1.5), we have and so either $p_1(z,STz) = p_1(STz,z) = 0$

or
$$p_1(STz,z) \leqslant Cp_1(z,STz).$$

Similarly, appling inequality (1.1) we get

$$p_1(z,STz) \leqslant Cp_1(STz,z)$$

from which it follows that

$$p_1(z,STz) = p_1(STz,z) = 0.$$

Since $p_1(z,z) = 0$, from lemma 1.2, $STz = z$. Hence we have $STz = Sw = z$.

Now suppose that ST has a second fixed point z'. Then By inequalities (1.1), We have

$p_1(z,z') = p_1(STz,STz') \leqslant C \max\{p_1(z,z'), p_1(z,z), p_1(z',z'), p_2(Tz,Tz')\}$

and so either $p_1(z,z') = 0$

or
$$p_1(z,z') \leqslant C \max\{p_1(z',z'), p_2(Tz,Tz')\}. \tag{1.6}$$

By inequalities (1.1), We have

$p_2(Tz',Tz') = p_2(TSTz',TSTz') \leqslant C \max\{p_2(Tz',Tz'), p_2(Tz',Tz'), p_2(Tz',Tz'), P_1(z',z')\}$

and so either $\quad p_2(Tz',Tz') = 0 \quad$ or $\quad p_2(Tz',Tz') \leqslant C\, p_1(z',z'). \qquad$ (1.7)

Applying inequality (1.1) we have

$p_1(z',z') = p_1(STz',STz') \leqslant C \max\{p_1(z',z'), p_1(z',z'), p_1(z',z'), p_2(Tz',Tz')\}$

and so either $\quad p_1(z',z') = 0 \quad$ or $\quad p_1(z',z') \leqslant C\, p_2(Tz',Tz'). \qquad$ (1.8)

By (1.7) and (1.8), it follows that

$$p_1(z',z') \leqslant C^2 p_1(z',z'),\quad p_2(Tz',Tz') \leqslant C^2 p_2(Tz',Tz').$$

Which implies that $p_1(z',z') = 0$, $p_2(Tz',Tz') = 0$.

By inequality (1.1), we have

$$\begin{aligned} p_2(Tz,Tz') &= p_2(TSTz,TSTz') \\ &\leqslant C \max\{p_2(Tz,Tz'), p_2(Tz,Tz), p_2(Tz',Tz'), p_1(STz,STz')\} \\ &= C \max\{p_2(Tz,Tz'), p_2(w,w), p_2(Tz',Tz'), p_1(z,z')\} \end{aligned}$$

and so either $p_2(Tz,Tz') = 0$ or $p_2(Tz,Tz') \leqslant C\, p_1(z,z')$.

If $p_2(Tz,Tz') = 0$, then $p_1(z,z') = 0$. If $p_2(Tz,Tz') \leqslant C\, p_1(z,z')$, since $p_1(z',z') = 0$, from inequality (1.6), we have $p_1(z,z') \leqslant C^2 p_1(z,z')$, which implies that $p_1(z,z') = 0$.

Since $p_1(z,z) = 0$, from lemma (1.2), We have $z = z'$.

Similarly, w is the unique fixed point of TS. This complete the poof of the theorem.

THEOREM2. 2. Let (X,d) and (Y,ρ) be complete metric spaces. Let p_1 be a w — distance on X, p_2 be a w — distance on Y. If T be a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities

$$p_1(Sy,Sy')p_1(STx,STx') \leqslant C \max\{p_1(Sy,Sy')p_2(Tx,Tx'),[p_1(Sy,x')]^2, p_1(x,x')p_1(Sy,Sy'),p_1(Sy,STx)p_1(Sy',STx')\} \quad (1.9)$$

$$p_2(Tx,Tx')p_2(TSy,TSy') \leqslant C \max\{p_1(Sy,Sy')p_2(Tx,Tx'),[p_2(Tx,y')]^2, p_2(y,y')p_2(Tx,Tx'),p_2(Tx,TSy)p_2(Tx',TSy')\} \quad (1.10)$$

for all x,x' in X and y,y' in Y, where $0 < C < 1$. Then

(1)For each $x \in X$, $\{(ST)^n x = x_n\}$ is a cauchy sequence with a limit z in X and $\{T(ST)^{n-1}x = y_n\}$ is a cauchy sequence with a limit w in Y.

(2)$p_1(z,z) = 0$ and $p_2(w,w) = 0$

(3)If either T or S is continuous then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

PROOF. Let x be an arbitrary point in X. We define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$$(ST)^n x = x_n, \quad T(ST)^{n-1}x = y_n$$

for $n = 1,2,\cdots$.

We will assume tha $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n, otherwise, if $x_n = x_{n+1}$ and $y_n = y_{n+1}$ for some n, We could put $x_n = z$ and $y_n = w$.

Applying inequality (1. 9) we get

$$p_1(x_{n-1},x_n)\, p_1(x_n,x_{n+1}) = p_1(Sy_{n-1},Sy_n)p_1(STx_{n-1},STx_n) \leqslant C \max\{p_1(x_{n-1},x_u)\, p_2(y_n, y_{n+1}),[p_1(x_{n-1},x_n)]^2,\ p_1(x_{n-1},x_n)p_1(x_{n-1},x_n),p_1(x_{n-1},x_n)p_1(x_n,x_{n+1})\}$$

Since $0 < C < 1$ and $x_n \neq x_{n+1}$ $y_n \neq y_{n+1}$, from which it follows that

$$p_1(x_n,x_{n+1}) \leqslant C \max\{p_2(y_n,y_{n+1}),p_1(x_{n-1},x_n)\}$$

Applying inequality (1. 10) we get

$$p_2(y_{n-1},y_n)p_2(y_n,y_{n+1}) = p_2(Tx_{n-2},Tx_{n-1})p_2(TSy_{n-1},TSy_n)$$

$$\leqslant C \max\{p_1(x_{n-1},x_n)p_2(y_{n-1},y_n),[p_2(y_{n-1},y_n)]^2,[p_2(y_{n-1},y_n)]^2,p_2(y_{n-1},y_n)p_2(y_n,y_{n+1})\}$$

Since $0 < C < 1$ and $x_n \neq x_{n+1}$, $y_n \neq y_{n+1}$, from which it follws that

$$p_2(y_n,y_{n+1}) \leqslant C \max\{p_1(x_{n-1},x_n),p_2(y_{n-1},y_n)\}$$

It now follows easily by induction that

$$p_1(x_n,x_{n+1}) \leqslant C^n \max\{p_1(x,x_1),p_2(y_1,y_2)\}$$

$$p_2(y_n,y_{n+1}) \leqslant C^{n-1}\max\{p_1(x,x_1),p_2(y_1,y_2)\}$$

for $n = 1,2,\cdots$. If $n < m$, then

$$p_1(x_n,x_m) \leqslant p_1(x_n,x_{n+1}) + p_1(x_{n+1},x_{n+2}) + \cdots + p_1(x_{m-1},x_m))$$
$$\leqslant (C^n + C^{n+1} + \cdots + C^{m-1})\max\{p_1(x,x_1),p_2(y_1,y_2)\}$$
$$\leqslant \frac{C^n}{1-C}\max\{P_1(x,x_1),P_2(y_1,y_2)\} \to 0(n \to \infty)$$
$$p_2(y_n,y_m) \leqslant p_2(y_n,y_{n+1}) + p_2(y_{n+1},y_{n+2}) + \cdots + p_2(y_{m-1},y_m)$$
$$\leqslant (C^{n-1} + C^n + \cdots + C^{m-2})\max\{p_1(x,x_1),p_2(y_1,y_2)\}$$
$$\leqslant \frac{C^{n-1}}{1-C}\max\{p_1(x,x_1),p_2(y_1,y_2)\} \to 0(n \to \infty)$$

Since $0<C<1$, from lemma 1. 2, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limis z in X with w in Y.

(2)Applying inequality (1. 9) we get

$$p_1(x_n,x_n)p_1(x_n,x_n) = p_1(Sy_n,Sy_n)p_1(STx_{n-1},STx_{n-1})$$
$$\leqslant C\max\{p_1(x_n,x_n)p_2(y_n,y_n),\ [p_1(x_n,x_{n-1})]^2,p_1(x_{n-1},x_{n-1})p_1(x_n,x_n),p_1(x_n,x_n)p_1(x_n,x_n)\}$$

Letting n tend to infinity, we have

$$[p_1(z,z)]^2 \leqslant C\max\{\ p_1(z,z)p_2(w,w),[p_1(z,z)^2]^2\}$$

and so either $p_1(z,z) = 0$ or

$$p_1(z,z) \leqslant C\ p_2(w,w) \tag{1. 11}$$

Applying inequality (1. 10) we get

$$p_2(y_n,y_n)p_2(y_n,y_n) = p_2(Tx_{n-1},Tx_{n-1})p_2(TSy_{n-1},TSy_{n-1})$$
$$\leqslant C\max\{p_1(x_{n-1},x_{n-1})p_2(y_n,y_n),[p_2(y_n,y_{n-1})]^2,p_2(y_{n-1},y_{n-1})p_2(y_n,y_n),[p_2(y_n,y_n)]^2\}$$

Letting n tend to infinity, we have

$$[p_2(w,w)]^2 \leqslant C\max\{p_1(z,z)p_2(w,w),[p_2(w,w)]^2\}$$

and so either $p_2(w,w) = 0$ or

$$p_2(w,w) \leqslant C\ p_1(z,z) \tag{1. 12}$$

From inequality (1. 11) and (1. 12), we get

$$p_1(z,z) \leqslant C^2p_1(z,z),\ p_2(w,w) \leqslant C^2p_2(w,w)$$

from with it follows that $p_1(z,z) = 0$, $p_2(w,w) = 0$.

(3)Applying inequality (1. 9) we get

$$p_1(x_n,Sw)p_1(x_{n+1},STz) = p_1(Sy_n,Sw)p_1(STx_n,STz)$$
$$\leqslant C\max\{p_1(x_n,Sw)p_2(y_{n+1},Tz),[p_1(x_n,z)]^2,p_1(x_n,z)p_1(x_n,Sw),p_1(x_n,x_{n+1})p_1(Sw,STz)\}$$

Letting n tend to infinity, we have

$$p_1(z,Sw)p_1(z,STz) \leqslant C\ p_1(z,Sw)p_2(w,Tz)$$

and so either $$p_1(z,Sw) = 0 \tag{1.13}$$

or $$p_1(z,STz) \leqslant C\, p_2(w,Tz) \tag{1.14}$$

Applying inequality (1.10) we have

$p_2(y_{n+1},Tz)p_2(y_{n+1},TSw) = p_2(Tx_n,Tz)p_2(Tsy_n,TSw)$

$\leqslant C \max\{p_1(x_n,Sw)p_2(y_{n+1},Tz),\ [p_2(y_{n+1},w)]^2, p_2(y_n,w)p_2(y_{n+1},Tz), p_2(y_{n+1},y_{n+1})p_2(Tz,Tsw)\}$

Letting n tend to infinity, we have

$$p_2(w,Tz)p_2(w,Tsw) \leqslant C\, p_1(z,Sw)p_2(w,Tz)$$

and so either $$p_2(w,Tz) = 0 \tag{1.15}$$

or $$p_2(w,TSw) \leqslant C\, p_1(z,Sw) \tag{1.16}$$

(i)If T is continuous, then

$$w = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} Tx_n = Tz.$$

(a)If equation (1.13) holds ,since $p_1(z,z) = 0$, by lemma 1.2 we have $Sw = z$. We then have $TSw = Tz = w$ and $STz = Sw = z$.

(ii)If inequality (1.14) holds, since $p_2(w,Tz) = p_2(w,w) = 0$. hence $p_1(z,STz) = 0$, Also since $p_1(z,z) = 0$, from lemma 1.2 we have $STz = z$ and $Sw = z$, $TSw = Tz = w$.

(2)If S is continuous, then

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Sy_n = Sw.$$

(a)If equation (1.15) holds, since $p_2(w,w) = 0$, by lemma 1.2 we have $w = Tz$. We then have $Sw = STz = z$ and $TSw = Tz = w$.

(b)If inequality (1.16) holds, since $p_1(z,Sw) = p_1(z,z) = 0$, hence $p_2(w,TSw) = 0$. Also since $p_2(w,w) = 0$, from lemma 1.2 We have $w = TSw = Tz$ and $STz = Sw = z$.

To prove uniqueness, suppose that ST has a second fixed point z' and TS has a second fixed point w'. Then applying inequality (1.9) we have

$[p_1(z,z')]^2 = p_1(STz,STz')p_1(STz,STz')$

$\leqslant C \max\{p_1(z,z')p_2(Tz,Tz'),[p_1(z,z')]^2,p_1(z,z')p_1(z,z'),p_1(z,z)p_1(z',z')\}$

$= C \max\{p_1(z,z')p_2(Tz,Tz'),[p_1(z,z')]^2\}$

and so either $p_1(z,z') = 0$ (1.17)

or $$p_1(z,z') \leqslant C\, p_2(Tz,Tz') \tag{1.18}$$

Further applying inequality (1.10) we have

$$[p_2(Tz,Tz')]^2 = p_2(Tz,Tz')p_2(TSTz,TSTz')$$

$\leqslant C \max\{p_1(z,z')p_2(Tz,Tz'),[p_2(Tz,Tz')]^2,[p_2(Tz,Tz')]^2,p_2(Tz,Tz)p_2(Tz',Tz')\}$

and so either $$p_2(Tz,Tz') = 0 \tag{1.19}$$

$$p_2(Tz,Tz') \leqslant C\, p_1(z,z') \tag{1.20}$$

If equation (1.17) holds, since $p_1(z,z) = 0$, from lemma 1.2, we have $z = z'$.

If inequality (1.18) holds, then

(a)If equality (1.19) hold, then $p(z,z') = 0$. We then have $z = z'$.

(b)If inequality (1.20) hold, then $p_1(z,z') \leqslant C\, p_2(Tz,Tz') \leqslant C^2 p_1(z,z')$ and so $p_1(z,z') = 0$ Since $0 < C < 1$. Since $p_1(z,z) = 0$, from lemma 1.2, we have $z = z'$, proving the uniqueness of the fixed point z of ST.

Now $TSw' = w'$ implies $STSw' = Sw'$ and so $Sw' = z$. Thus

$$w = TSw = Tz = TSw' = w'$$

proving that w is the unique fixed point of TS. This completes the proof of the theorem.

COROLLARY2.3. Let (X,d) be a complete metric space and p be a w — distance on X. If T be a mapping of X into X Satisfying the inequality

$$p(Ty,Ty')p(T^2x,T^2x') \leqslant C\max\{p(Ty,Ty')p(Tx,Tx'),[p(Ty,x')]^2,p(x,x')p(Ty,Ty'),p(Ty,T^2x)p(Ty',T^2x')\}$$

for all x, x', y, y' in X. Where $0 < C < 1$, Then T has a unique fixed point z in X.

PROOF. It follows from theorem with $(X,d) = (Y,\rho)$ and $S = T$, $p_1 = p_2$ that T^2 has a unique fixed point z. Then $T^2(Tz) = T(T^2z) = Tz$ and so we see that Tz is also a fixed point of T^2. Since the fixed point is unique, we nust have $Tz = z$.

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