Generalized intuitionistic fuzzy sets

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Abstract : In this paper we introduce definitions of generalized intuitionistic fuzzy sets, generalized intuitionistic fuzzy relation, generalized intuitionistic fuzzy topology and study some of their properties.

Keywords : Fuzzy subset, intuitionistic fuzzy set, generalized intuitionistic fuzzy set, fuzzy relation, intuitionistic fuzzy family, generalized intuitionistic fuzzy topology.

0. Introduction

Let X be a nonempty set. In 1965, Zadeh [16] introduced the idea of fuzzy subset λ of X as a mapping $\lambda : X \to I$(the closed interval [0,1]). Later many authors generalized the idea of fuzzy subset in different directions. In [1] & [2], Atanassov introduced the concept of intuitionistic fuzzy set $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ where $\mu_A : X \to I$ and $\nu_A : X \to I$ are such that $\mu_A(x) + \nu_A(x) \leq 1,\ \forall\ x \in X$. Atanassov himself and many other authors (see [4], [7], [10], [11] etc.) studied different properties in intuitionistic fuzzy setting.

In this paper we introduce a generalized type of intuitionistic fuzzy set and derive various results in this setting.

1. Generalized intuitionistic fuzzy sets

Definition 1.1 Let X be a nonempty set. A generalized intuitionistic fuzzy set (GIF) A on X is an object having the form

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \qquad (*)$$

where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ satisfy the condition

$$\mu_A(x) \wedge \nu_A(x) \leq 0.5,\ \forall\ x \in X \qquad (**)$$

For each $x \in X$, $\mu_A(x)$ and $\nu_A(x)$ are called the degree of membership and the degree of non-membership, respectively, of x to A. The condition (**) is called the generalized intuitionistic condition (GIC).

For simplicity, we shall use $A = (\mu_A, \nu_A)$ in place of (*).
The collection of all GIFs on X is denoted by $\mathcal{C}(X)$.

Basic algebraic operations on $\mathcal{C}(X)$

Let $A, B, A_i \in \mathcal{C}(X)$, $\forall\ i \in I$. Then inclusion, equality, complementation, arbitrary union and arbitrary intersection on $\mathcal{C}(X)$ is defined as follows :
(1) $A \subset B \Leftrightarrow \mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$, $\forall\ x \in X$,
(2) $A = B \Leftrightarrow A \subset B$ and $B \subset A$,
(3) $\bar{A} = (\nu_A, \mu_A)$,
(4) $\cup_i A_i = (\vee_i \mu_{A_i}, \wedge_i \nu_{A_i})$,
(5) $\cap_i A_i = (\wedge_i \mu_{A_i}, \vee_i \nu_{A_i})$.

Definition 1.2 A generalized intuitionistic fuzzy point (GIFP) P on X is a GIF such that $\exists$ an $x \in X$ satisfying $\mu_P(x) > 0$ and $\mu_P(y) = 0$, $\nu_P(y) = 1$, $\forall\ y (\neq x) \in X$.
Such a GIFP is denoted by P_x. If for a GIFP P_x, $\mu_P(x) = a$ and $\nu_P(x) = b$, then the P_x is also denoted by $(a, b)_x$.
Let $A \in \mathcal{C}(X)$. Then the GIFP P_x is said to belong to A if $\mu_P(x) \le \mu_A(x)$ and $\nu_P(x) \ge \nu_A(x)$. This is denoted, symbolically, by $P_x \tilde{\in} A$.

Theorem 1.3 $A = \cup\{P_x\ :\ P_x \tilde{\in} A\}$, $\forall\ A \in \mathcal{C}(X)$.

Example 1.4 Let $X = \{a, b, c\}$. Then $A = \{(a, 0.8, 0.4), (b, 0.3, 0.9), (c, 0.6, 0)\}$ is a GIF on X. But A is not an intuitionistic fuzzy set on X.

Definition 1.5 We define $\tilde{0} = \widetilde{(0,1)}$ and $\tilde{1} = \widetilde{(1,0)}$.

Theorem 1.6 *For all $A, B, C, B_i \in \mathcal{C}(X)$, $i \in I$, we have*
(1) $\tilde{0} \subset A \subset \tilde{1}$,
(2) $\bar{(\tilde{0})} = \tilde{1}$; $\bar{(\tilde{1})} = \tilde{0}$,
(3) $A \subset B$ *and* $B \subset C \Rightarrow A \subset C$,
(4) $A, B \subset A \cup B$; $A, B \supset A \cap B$,
(5) $A \cup B = B \cup A$; $A \cap B = B \cap A$,
(6) $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$,
(7) $A \cup (\cap_i B_i) = \cap_i (A \cup B_i)$; $A \cap (\cup_i B_i) = \cup_i (A \cap B_i)$,
(8) $A \subset B \Leftrightarrow \bar{A} \supset \bar{B}$,
(9) $\overline{(\cup_i B_i)} = \cap_i \bar{B}_i$; $\overline{(\cap_i B_i)} = \cup_i \bar{B}_i$,
(10) $\bar{\bar{A}} = A$.

Definition 1.7 Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Let $A \in \mathcal{C}(X)$. Then the image of A, under f, denoted by $f(A) = (\mu_{f(A)}, \nu_{f(A)})$, is defined by

$$\mu_{f(A)}(y) = \begin{cases} \vee\{\mu_A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$\nu_{f(A)}(y) = \begin{cases} \wedge\{\nu_A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{otherwise} \end{cases}$$

Let $B \in \mathcal{C}(Y)$. Then the preimage of B, under f, denoted by $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$, is defined by
$\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$.

Theorem 1.8 *Let $A, A_i \in \mathcal{C}(X)$ and $B, B_j \in \mathcal{C}(Y)$, $i \in I, j \in J$ and $f : X \to Y$ be a function. Then*
(a) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$,
(b) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$,
(c) $f(\bar{A}) \supset \overline{[f(A)]}$, *if* f *is surjective,*
(d) $f^{-1}(\bar{B}) = \overline{[f^{-1}(B)]}$,
(e) $A \subset f^{-1}(f(A))$, *the equality holds if f is injective,*
(f) $f(f^{-1}(B)) \subset B$, *the equality holds if f is surjective,*
(g) $f^{-1}(\cup_j B_j) = \cup_j f^{-1}(B_j)$,
(h) $f^{-1}(\cap_j B_j) = \cap_j f^{-1}(B_j)$,
(i) $f(\cup_i A_i) = \cup_i f(A_i)$,
(j) $f(\cap_i A_i) \subset \cap_i f(A_i)$, *the equality holds if f is injective,*
(k) If $g : Y \to Z$ *be a mapping such that* $g^{-1} : Z \to Y$ *then* $(gof)^{-1}(C) = f^{-1}(g^{-1}(C))$, *for any* $C \in \mathcal{C}(Z)$, gof *is the composition of* g *and* f.

2. Generalized intuitionistic fuzzy relations

Let X, Y and Z be three ordinary nonempty sets.
Definition 2.1 A generalized intuitionistic fuzzy relation is defined as a generalized intuitionistic fuzzy subset of $X \times Y$, having the form
$R = \{< (x, y), \mu_R(x, y), \nu_R(x, y) > \; : \; x \in X, \; y \in Y\}$
where $\mu_R : X \times Y \to [0, 1]$, $\nu_R : X \times Y \to [0, 1]$ satisfy the condition $\mu_R(x, y) \wedge \nu_R(x, y) \leq 0.5$, $\forall \; (x, y) \in X \times Y$.
The collection of all GIFRs on $X \times Y$ is denoted by $GR(X \times Y)$.

Definition 2.2 Let R be a GIFR on $X \times Y$. Then we define inverse relation of R, denoted by R^{-1}, by

$$\mu_{R^{-1}}(y, x) = \mu_R(x, y), \; \nu_{R^{-1}}(y, x) = \nu_R(x, y), \; \forall \; (x, y) \in X \times Y.$$

Definition 2.3 Let $P, Q \in GR(X \times Y)$. Then for every $(x, y) \in X \times Y$ we define
(a) $P \leq Q \Leftrightarrow \mu_P(x,y) \leq \mu_Q(x,y)$ and $\nu_P(x,y) \geq \nu_Q(x,y)$,
(b) $P \lessapprox Q \Leftrightarrow \mu_P(x,y) \leq \mu_Q(x,y)$ and $\nu_P(x,y) \leq \nu_Q(x,y)$,
(c) $P \cup Q = \{< \mu_P(x,y) \vee \mu_Q(x,y), \nu_P(x,y) \wedge \nu_Q(x,y) > \; : \; (x,y) \in X \times Y\}$,
(d) $P \cap Q = \{< \mu_P(x,y) \wedge \mu_Q(x,y), \nu_P(x,y) \vee \nu_Q(x,y)) > \; : \; (x,y) \in X \times Y\}$,
(e) $\bar{P} = \{< \nu_P(x,y), \mu_P(x,y) > \; : \; (x,y) \in X \times Y\}$.

Theorem 2.4 *Let $P, Q, R \in GR(X \times Y)$. Then*
(a) $P \leq Q \Rightarrow R^{-1} \leq P^{-1}$,
(b) $(R \cup P)^{-1} = R^{-1} \cup P^{-1}$,
(c) $(R \cap P)^{-1} = R^{-1} \cap P^{-1}$,
(d) $(P^{-1})^{-1} = P$,
(e) $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$; $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$,
(f) $P \cup Q \geq P, Q$; $P \cap Q \leq P, Q$,
(g) *If* $P \geq Q$ *and* $P \geq R$ *then* $P \geq Q \cup R$; *if* $P \leq Q$ *and* $P \leq R$ *then* $P \leq Q \cap R$.

Definition 2.5 Let $R \in GR(X \times Y)$ and $P \in GR(Y \times Z)$. Then we define composed relation on $X \times Z$, denoted by PoR, by
$PoR = \{< (x,z), \mu_{PoR}(x,z), \nu_{PoR}(x,z) >: x \in X, z \in Z\}$, where
$\mu_{PoR}(x,z) = \vee_y\{\mu_R(x,y) \wedge \mu_P(y,z)\}$, $\nu_{PoR}(x,z) = \wedge_y\{\nu_R(x,y) \vee \nu_P(y,z)\}$.

Definition 2.6 Let $R \in GR(X \times Y)$ and $P \in GR(Y \times Z)$. Then we define another composed relation on $X \times Z$, denoted by $P * R$, by
$P * R = \{< (x,z), \mu_{P*R}(x,z), \nu_{P*R}(x,z) >: x \in X, z \in Z\}$, where
$\mu_{P*R}(x,z) = \wedge_y\{\mu_R(x,y) \vee \mu_P(y,z)\}$, $\nu_{P*R}(x,z) = \vee_y\{\nu_R(x,y) \wedge \nu_P(y,z)\}$.

Definition 2.7 Let $P, R \in GR(X \times X)$. Then P and R are said to commute if $PoR = RoP$.

Theorem 2.8 *For* $R \in GR(X \times Y)$, $P \in GR(Y \times Z)$, $(PoR)^{-1} = R^{-1}oP^{-1}$ *holds.*

Theorem 2.9 *If* $R, R_i \in GR(X \times Y)$ *and* $P, P_i \in GR(Y \times Z)$, $i = 1, 2$, *then*
(a) $P_1 \leq P_2 \Rightarrow P_1oR \leq P_2oR$,
(b) $R_1 \leq R_2 \Rightarrow PoR_1 \leq PoR_2$,
(c) $P_1 \lessapprox P_2 \Rightarrow P_1oR \lessapprox P_2oR$,
(d) $R_1 \lessapprox R_2 \Rightarrow PoR_1 \lessapprox PoR_2$,
(e) *If* $R, P \in GR(X \times X)$ *and* $P \leq R$ *then* $PoP \leq RoR$.

Theorem 2.10 *For* $R \in GR(X \times Y)$, $Q \in GR(Y \times Z)$ *and* $P \in GR(Z \times U)$
$(PoQ)oR = Po(QoR)$ *holds.*

Theorem 2.11 *For each* $R \in GR(X \times Y)$ *and* $P_i \in GR(Y \times Z), i \in I$,
(a) $(\cup_i P_i)oR = \cup_i(P_ioR)$,
(b) $(\cap_i P_i)oR = \cap_i(P_ioR)$

holds.

Here we shall define reflexivity, antireflexivity and study some of their properties.

Definition 2.12 Let $R \in GR(X \times X)$. Then

(a) R is reflexive of type-1 if

$\mu_R(x,x) = 1,\ \nu_R(x,x) = 0,\ \forall\, x \in X.$

(b) R is reflexive of type-2 if

$\mu_R(x,x) = 1,\ \forall\, x \in X,\ \nu_R(x,x) \vee \nu_R(y,y) \leq \nu_R(x,y),\ \forall\, x,y \in X.$

(c) R is reflexive of type-3 if

$\mu_R(x,x) \wedge \mu_R(y,y) \geq 0.5 \vee \mu_R(x,y),\ \forall\, x,y \in X,\ \nu_R(x,x) = 0,\ \forall\, x \in X.$

(d) R is reflexive of type-4 if

$\mu_R(x,x) \wedge \mu_R(y,y) \geq \mu_R(x,y),\ \nu_R(x,x) \vee \nu_R(y,y) \leq \nu_R(x,y),\ \forall\, x,y \in X.$

Definition 2.13 Let $R \in GR(X \times X)$. Then

(a) R is antireflexive of type-1 if

$\mu_R(x,x) = 0,\ \nu_R(x,x) = 1,\ \forall\, x \in X.$

(b) R is antireflexive of type-2 if

$\mu_R(x,x) = 0,\ \forall\, x \in X,\ \nu_R(x,x) \wedge \nu_R(y,y) \geq 0.5 \vee \nu_R(x,y),\ \forall\, x,y \in X.$

(c) R is antireflexive of type-3 if

$\mu_R(x,x) \vee \mu_R(y,y) \leq \mu_R(x,y),\ \forall\, x,y \in X,\ \nu_R(x,x) = 1,\ \forall\, x \in X.$

(d) R is antireflexive of type-4 if

$\mu_R(x,x) \vee \mu_R(y,y) \leq \mu_R(x,y),\ \nu_R(x,x) \wedge \nu_R(y,y) \geq \nu_R(x,y),\ \forall\, x,y \in X.$

Theorem 2.14

(a) Reflexivity (antireflexivity) of type-1 $\Rightarrow$reflexivity (antireflexivity) of type-2, 3 and 4,

(b) Reflexivity (antireflexivity) of type-2 $\Rightarrow$ reflexivity (antireflexivity) of type-4,

(c) Reflexivity (antireflexivity) of type-3 $\Rightarrow$ reflexivity (antireflexivity) of type-4.

Theorem 2.15

(a) If $R \in GR(X \times X)$ is reflexive of any type then $R \leq RoR$,

(b) If $R \in GR(X \times X)$ is antireflexive of any type then $R \geq R * R$.

Next we give examples of GIFRs which satisfy the property $R \leq RoR$ ($R \geq R * R$), but R is not reflexive (antireflexive) of any type.

Example 2.16 (a) Let $X = \{a,b,c\}$ and $R \in GR(X \times X)$ be given by

$$\mu_R = \begin{pmatrix} & a & b & c \\ a & 0.4 & 0.6 & 0.1 \\ b & 0.6 & 0.7 & 0.6 \\ c & 0.1 & 0 & 0.3 \end{pmatrix},\quad \nu_R = \begin{pmatrix} & a & b & c \\ a & 0.3 & 0.5 & 0.9 \\ b & 0.3 & 0.3 & 0.3 \\ c & 0.5 & 0.3 & 0.6 \end{pmatrix}$$

Then $R \leq RoR$, but R is not reflexive of any type.

(b) Let $X = \{a, b, c\}$ and $R \in GR(X \times X)$ be given by

$$\mu_R = \begin{pmatrix} & a & b & c \\ a & 0.4 & 0.6 & 0.3 \\ b & 0.6 & 0.7 & 0.6 \\ c & 0.3 & 0.4 & 0.3 \end{pmatrix}, \quad \nu_R = \begin{pmatrix} & a & b & c \\ a & 0.3 & 0.5 & 0.6 \\ b & 0.3 & 0.3 & 0.3 \\ c & 0.5 & 0.5 & 0.6 \end{pmatrix}$$

Then $R \geq R * R$, but R is not antireflexive of any type.

Theorem 2.17 *Let $R, R_1, R_2 \in GR(X \times X)$. Then*
(a) If R is reflexive (antireflexive) of any type, then RoR $(R * R)$ is reflexive (antireflexive) of the same type,
(b) If R is reflexive (antireflexive) of the type-i, then R^{-1} is reflexive (antireflexive) of the type, i = 1, 2, 3 and 4,
(c) If both R_1 and R_2 are reflexive (antireflexive) of the type-i, then $R_1 \wedge R_2$ $(R_1 \vee R_2)$ is reflexive (antireflexive) of the type-i, i = 1, 2, 3 and 4,
(d) If R_1 is reflexive (antireflexive) of the type-1, then $R_1 \vee R_2$ $(R_1 \wedge R_2)$ is so. If R_1 and R_2 are reflexive (antireflexive) of the type-i, then $R_1 \vee R_2$ $(R_1 \wedge R_2)$ is reflexive (antireflexive) of the type-i, i = 2, 3 and 4.

Definition 2.18 A relation $R \in GR(X \times X)$ is called symmetric if $R = R^{-1}$ i.e., if for all $(x, y) \in X \times X$, $\mu_R(x, y) = \mu_R(y, x)$, $\nu_R(x, y) = \nu_R(y, x)$.

Theorem 2.19
(a) If $P, R \in GR(X \times X)$ are symmetrical, then $PoR = (RoP)^{-1}$,
(b) If R is symmetrical then RoR is symmetrical.

Definition 2.20
(1) We define $R \in GR(X \times X)$ to be transitive if $R \geq RoR$,
(2) We define $R \in GR(X \times X)$ to be c-transitive if $R \leq R * R$.

Definition 2.21 Let $R \in GR(X \times X)$.
(a) The transitive closure of R is defined to be the minimum GIFR $\hat{R}$ on $X \times X$ which contains R and it is transitive, that is to say
(1) $R \leq \hat{R}$,
(2) $\hat{R}o\hat{R} \leq \hat{R}$,
(3) if $P \in GR(X \times X)$, $R \leq P$ and P is transitive, then $\hat{R} \leq P$.
(b) The c-transitive closure of R is defined to be the biggest c-transitive relation $\check{R} \in GR(X \times X)$ contained in R.

Notation 2.22 We denote $R^1 = R$, $R^n = RoRo......n$ times, $n \geq 2$ and $R^{*1} = R$, $R^{*n} = R * R * ....n$ times, $n \geq 2$.

Theorem 2.23 *For every $R \in GR(X \times X)$, it is verified that :*

(a) $\hat{R} = R^1 \vee R^2 \vee R^3 \vee ...... \vee R^n \vee ...... = \vee_{i=1}^{\infty} R^i$,
(b) $\breve{R} = R^{*1} \wedge R^{*2} \wedge R^{*3} \wedge ...... \wedge R^{*n} \wedge ...... = \wedge_{i=1}^{\infty} R^{*i}$.

Theorem 2.24 *Let $R,\ P \in GR(X \times X)$. Then $R \leq P \Rightarrow \hat{R} \leq \hat{P}$ and $\breve{R} \geq \breve{P}$.*

Corollary 2.25 *For every $R \in GR(X \times X)$, $\breve{R} \leq R \leq \hat{R}$ holds.*

Corollary 2.26
(1) If $R \in GR(X \times X)$ is reflexive of any type and transitive then $R = RoR$.
(2) If $R \in GR(X \times X)$ is antireflexive of any type and c-transitive then $R = R * R$.

3. Generalized intuitionistic fuzzy families

Šhostak [14] introduced the idea of fuzzy family and gave definitions of union, intersection of a fuzzy family. Later Çoker [8] defined intuitionistic fuzzy family (IFF) and studies properties related to union, intersection, complement and functional image of IFF. In this section we give the definition of generalized intuitionistic fuzzy family (GIFF) and study various properties of GIFF involving union, intersection etc.

Definition 3.1 A GIF G on the set $\mathcal{C}(X)$ is called a generalized intuitionistic fuzzy family (GIFF) on X.

Definition 3.2 A GIFF $\mathcal{B}$ on X is said to be a finite GIFF if for some positive integer n, $\exists\ B_1, B_2, ..., B_n \in \mathcal{C}(X)$ s.t.
$\mu_{\mathcal{B}}(B_i) > 0,\ 1 \leq i \leq n$
and
$\mu_{\mathcal{B}}(A) = 0,\ \nu_{\mathcal{B}}(A) = 1,$ if $A \in \mathcal{C}(X) \setminus \{B_1, B_2, ..., B_n\}$.

If $\mu_{\mathcal{B}}(B_i) = p_i$ and $\nu_{\mathcal{B}}(B_i) = q_i,\ 1 \leq i \leq n$, then $\mathcal{B}$ is expressed as $\mathcal{B} = \{\frac{B_1}{(p_1,q_1)}, ..., \frac{B_n}{(p_n,q_n)}\}$.

Notation 3.3 Let G be a GIFF on X. We denote
$S(G) = \{\lambda \in \mathcal{C}(X)\ :\ \mu_G(\lambda) > 0\}$ and $S^*(G) = \{\lambda \in \mathcal{C}(X)\ :\ \nu_G(\lambda) < 1\}$.

Definition 3.4 Let G be a GIFF on X. Then the GIFF $G^* = (\mu_{G^*}, \nu_{G^*})$ defined by $\mu_{G^*}(A) = \mu_G(\bar{A})$ and $\nu_{G^*}(A) = \nu_G(\bar{A})$, $\forall A \in \mathcal{C}(X)$, is called the complement of G.

Definition 3.5 Let G be a GIFF on X. Then the union $P = \cup G$ and the intersection $Q = \cap G$ of this GIFF are defined as follows :
(a) $P = \cup G$ is the GIF $P = (\mu_P, \nu_P)$, where
$\mu_P(x) = \vee_{A \in \mathcal{C}(X)}\{\mu_G(A) \wedge \mu_A(x)\},\ \nu_P(x) = \wedge_{A \in \mathcal{C}(X)}\{\nu_G(A) \vee \nu_A(x)\},\ \forall\ x \in X$.
(b) $Q = \cap G$ is the GIF $Q = (\mu_Q, \nu_Q)$, where
$\mu_Q(x) = \wedge_{A \in \mathcal{C}(X)}\{\nu_G(A) \vee \mu_A(x)\},\ \nu_Q(x) = \vee_{A \in \mathcal{C}(X)}\{\mu_G(A) \wedge \nu_A(x)\},\ \forall\ x \in X$.

Theorem 3.6 *Let G be a GIFF on X. Then we have*

(a) $\overline{\cup G} = \cap G^*$,
(b) $\overline{\cap G} = \cup G^*$.

Let X and Y be two nonempty sets and $f : X \to Y$ be a function.

Definition 3.7 (a) Let G be a GIFF on X. If f is injective, then the image $f(G) = (\mu_{f(G)}, \nu_{f(G)})$ of G under f is defined by

$$\mu_{f(G)}(A) = \begin{cases} \mu_G(f^{-1}(A)) & \text{if } A \leq \tilde{1}_{f(X)}, \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{f(G)}(A) = \begin{cases} \nu_G(f^{-1}(A)) & \text{if } A \leq \tilde{1}_{f(X)}, \\ 1 & \text{otherwise} \end{cases}$$

where $\tilde{1}_{f(X)} = (\chi_{f(X)}, \chi_{Y-f(X)})$.

(b) Let H be a GIFF on Y. Then the preimage $f^{-1}(H) = (\mu_{f^{-1}(H)}, \nu_{f^{-1}(H)})$ of H under f is defined by
$\mu_{f^{-1}(H)}(B) = \vee\{\mu_H(C); B = f^{-1}(C), C \in \mathcal{C}(Y)\}$,
$\nu_{f^{-1}(H)}(B) = \wedge\{\nu_H(C); B = f^{-1}(C), C \in \mathcal{C}(Y)\}$, $B \in \mathcal{C}(X)$.

Theorem 3.8 *Let $f : X \to Y$ be a mapping. If H is a GIFF on Y, then*
(a) $f^{-1}(\cup H) = \cup f^{-1}(H)$,
(b) $f^{-1}(\cap H) = \cap f^{-1}(H)$.
If G is a GIFF on X and f is injective, then
(c) $f(\cup G) = \cup f(G)$,
(d) $f(\cap G) = \cap f(G)$.

4. Generalized intuitionistic fuzzy topology

In [12] and [13], Samanta et al. introduced the definition of gradation of openness of fuzzy subsets and thereby generalized the definition of fuzzy topology as introduced by Chang [6]. Independently, some other authors including Šhostak [14], Ramadan [9], Ying [15] etc. generalized the Chang's definition of fuzzy topology. In this section, we give the definition of generalized intuitionistic fuzzy topology (GIFT) by incorporating the idea of GIFF and using the ideas of union and intersection of GIFF. Perhaps in fuzzy setting also this idea of fuzzy topology (by using the concept of fuzzy family together with the ideas of union and intersection of a fuzzy family) is new.

Definition 4.1 A generalized intuitionistic fuzzy family τ on X is said to be a generalized intuitionistic gradation of openness (GIGO) on X if it satisfies the following conditions :
(1) $\mu_\tau(\tilde{0}) = \mu_\tau(\tilde{1}) = 1$, $\nu_\tau(\tilde{0}) = \nu_\tau(\tilde{1}) = 0$,
(2) for any GIFF $\mathcal{G}$ on X,
$\mu_\tau(\cup\mathcal{G}) \geq \wedge_{A \in S(\mathcal{G})}(\mu_\tau(A) \wedge \mu_{\mathcal{G}}(A))$,
$\nu_\tau(\cup\mathcal{G}) \leq \vee_{A \in S^*(\mathcal{G})}(\nu_\tau(A) \vee \nu_{\mathcal{G}}(A))$,
(3) for any finite GIFF $\mathcal{B} = \{\frac{B_1}{(p_1,q_1)}, \frac{B_2}{(p_2,q_2)}, ..., \frac{B_n}{(p_n,q_n)}\}$ in X,

$\mu_\tau(\cap\mathcal{B}) \geq \wedge_{i=1}^{n}(\mu_\tau(B_i) \wedge \mu_\mathcal{B}(B_i))$,
$\nu_\tau(\cap\mathcal{B}) \leq \vee_{i=1}^{n}(\nu_\tau(B_i) \vee \nu_\mathcal{B}(B_i))$.

If τ is a GIGO on X then τ is also called a generalized intuitionistic fuzzy topology (GIFT) on X and (X,τ) is called a generalized intuitionistic fuzzy topological space (GIFTS). μ_τ and ν_τ may be interpreted as generalized gradation of openness and generalized gradation of nonopenness, respectively.

In the definition of gradation of openness τ on X in fuzzy setting Samanta et al. [13] considered the conditions as
(i) $\mu_\tau(\cup_{i\in\Delta}\lambda_i) \geq \wedge_{i\in\Delta}\mu_\tau(\lambda_i)$,
(ii) $\mu_\tau(\cap_{i=1}^{n}\lambda_i) \geq \wedge_{i=1}^{n}\mu_\tau(\lambda_i)$,
where $\cup_{i=\Delta}\lambda_i$ and $\cap_{i=1}^{n}\lambda_i$ are respective union and intersection of the crisp collections $\{\lambda_i\ ,\ i \in \Delta\}$ and $\{\lambda_i\ ,\ i = 1,2,....,n\}$ of fuzzy subsets of X; instead in the present setting we have taken them GIFs $\mathcal{G}$ and $\mathcal{B}$ of $\mathcal{C}(X)$ respectively and in condition (2) and (3) of Definition 4.1 of a GIGO we have taken correspondingly union and intersection of $\mathcal{G}$ and $\mathcal{B}$ respectively involving the gradation of A ($\in \mathcal{C}(X)$) w.r.t. $\mathcal{G}$ and $\mathcal{B}$. Thus a definition of gradation of openness (GO) can be derived from the Definition 4.1 of a GIGO and in this definition of GO, the scope of dealing with the fuzziness in the fuzzy topology is improved.

Definition 4.2 A GIFF $\mathcal{F}$ on X is said to form a generalized intuitionistic gradation of closedness (GIGC) on X if it satisfies the following conditions :
(1) $\mu_\mathcal{F}(\tilde{0}) = \mu_\mathcal{F}(\tilde{1}) = 1,\ \nu_\mathcal{F}(\tilde{0}) = \nu_\mathcal{F}(\tilde{1}) = 0$,
(2) for any GIFF $\mathcal{G}$ on X,
$\mu_\mathcal{F}(\cap\mathcal{G}) \geq \wedge_{A\in S(\mathcal{G})}(\mu_\mathcal{F}(A) \wedge \mu_\mathcal{G}(A))$,
$\nu_\mathcal{F}(\cap\mathcal{G}) \leq \vee_{A\in S^*(\mathcal{G})}(\nu_\mathcal{F}(A) \vee \nu_\mathcal{G}(A))$,
(3) for any finite GIFF $\mathcal{B} = \{\frac{B_1}{(p_1,q_1)}, \frac{B_2}{(p_2,q_2)}, ..., \frac{B_n}{(p_n,q_n)}\}$ in X,
$\mu_\mathcal{F}(\cup\mathcal{B}) \geq \wedge_{i=1}^{n}(\mu_\mathcal{F}(B_i) \wedge \mu_\mathcal{B}(B_i))$,
$\nu_\mathcal{F}(\cup\mathcal{B}) \leq \vee_{i=1}^{n}(\nu_\mathcal{F}(B_i) \vee \nu_\mathcal{B}(B_i))$.

Theorem 4.3
(a) τ is a GIGO on X iff τ^* is a GIGC on X,
(b) $\mathcal{F}$ is a GIGC on X iff $\mathcal{F}^*$ is a GIGO on X.

Theorem 4.4 *Let (X,τ) be a GIFTS and $Y \subset X$. Let us define two mappings μ_{τ_Y}, ν_{τ_Y} : $\mathcal{C}(Y) \to I$ by the rule :*
$\mu_{\tau_Y}(\lambda) = \vee\{\mu_\tau(A) : A \in \mathcal{C}(X),\ A/Y = \lambda\},\ \nu_{\tau_Y}(\lambda) = \wedge\{\nu_\tau(A) : A \in \mathcal{C}(X),\ A/Y = \lambda\}$.
Then $\tau_Y = (\mu_{\tau_Y}, \nu_{\tau_Y})$ is a GIFT on Y.

Theorem 4.5 *Let (Y,τ_Y) be a subspace of the GIFTS (X,τ) and $\lambda \in \mathcal{C}(X)$. Then*
(a) $\mu_{\tau_Y^*}(\lambda) = \vee\{\mu_{\tau^*}(A)\ :\ A \in \mathcal{C}(X),\ A/Y = \lambda\},\ \nu_{\tau_Y^*}(\lambda) = \wedge\{\nu_{\tau^*}(A)\ :\ A \in \mathcal{C}(X),\ A/Y = \lambda\}$,

(b) if $Z \subset Y \subset X$, then $\mu_{\tau_Z} = (\mu_{\tau_Y})_Z$, $\nu_{\tau_Z} = (\nu_{\tau_Y})_Z$.

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