Semi-preconnectedness in fts *

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Abstract

We introduce a kind of new connectedness of fuzzy sets, that is fuzzy semi-preconnectedness, and establish some of its fundamental properties in fuzzy topological spaces.

Keywords: Fuzzy topology; Semi-preopen set; Semi-preconnectedness; SP-irresolute mapping

1. Introduction

The connectedness for fuzzy topological spaces have been defined in different ways by different researchers[2,4,6-8]. In this paper, we follow the concept of fuzzy semi-preopen set[9],with the help of fuzzy semi-pre-neighborhoods, to introduce a kind of new connectedness, that is fuzzy semi-preconnectedness, and also establish some of its fundamental properties in fuzzy topological spaces. At the same time, we discuss the relation between fuzzy semi-preconnected set and fuzzy semi-connected set.

2. Preliminaries

In this paper, by X and Y we mean fuzzy topological spaces(fts, for short). For two fuzzy sets A and B in X, we write AqB to mean that A is quasi-coincident

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with B. Negation of such a statement is denoted as $A \not{q} B$. B is said to be a quasi-neighbourhood of A iff there exists a fuzzy open set U such that $AqU \leq B$[7]. The constant fuzzy sets taking on the values 0 and 1 on X are designated by 0_X and 1_X respectively. For a fuzzy set A in X, the notations A^o, A^-, A' and $suppA$ will respectively stand for the interior, closure, complement and support of A. A fuzzy set A in X is said to be (1) fuzzy semi-preopen if there is a fuzzy preopen set B such that $B \leq A \leq B^-$; (2) fuzzy semi-preclosed if there is a fuzzy preclosed set B such that $B^o \leq A \leq B$[9]. The family of fuzzy semi-preopen (resp. semi-preclosed) sets of a fts X will be denoted by $\psi(X)(resp.\psi'(X))$. $A^{\square} = \bigcup\{B : B \leq A, B \in \psi(X)\}$ and $A^{\frown} = \bigcap\{B : A \leq B, B \in \psi'(X)\}$ are called the semi-preinterior and semi-preclosure of fuzzy set A, respectively. A fuzzy set A is called a fuzzy semi-preneighborhood of a fuzzy point x_α in X if there exists a $B \in \psi(X)$ such that $x_\alpha \in B \leq A$. A fuzzy set A is called a fuzzy semi-pre-q-neighborhood of a fuzzy point x_α in X if there exists a $B \in \psi(X)$ such that $x_\alpha qB \leq A$[9]. The set of all semi-preneighborhoods (semi-pre-q-neighborhoods) of x_α is denoted by $\xi(x_\alpha)(\eta(x_\alpha))$. We easily prove that $A^{\square\prime} = A^{\prime\frown}$ and $A^{\frown\prime} = A^{\prime\square}$ for a fuzzy set A in X.

A mapping $f : X \to Y$ is called fuzzy SP-irresolute if $f^{-1}(B) \in \psi(X)$ for each $B \in \psi(Y)$[3].

3. Fuzzy semi-pre-connectedness

Definition 3.1. Two non-null fuzzy sets A and B in an fts X are said to be fuzzy semi-preseparated iff $A \not{q} B^{\frown}$ and $B \not{q} A^{\frown}$.

Definition 3.2. A fuzzy set which cannot be expressed as the union of two fuzzy semi-preseparated sets is said to be a fuzzy semi-preconnected set.

Remark 3.3. (1) Clearly, if fuzzy sets A and B in an fts X are fuzzy semi-separated[6] then they are fuzzy semi-preseparated.

(2) Clearly, every fuzzy semi-preconnected set is fuzzy semi-connected[6]. That the converses of (1) and (2) need not be true is shown by the following Example 3.4.

Example 3.4. Let $X = [0,1]$ and A, B, C be fuzzy sets in X difined as follows:

$$A(x) = \begin{cases} 0.5, & if\ x = 0, \\ 0, & if\ 0 < x \leq 1; \end{cases}$$

$$B(x) = \begin{cases} 0.7, & if\ x = 0, \\ 1, & if\ 0 < x \leq 1; \end{cases}$$

$$C(x) = \begin{cases} 0.1, & if\ x = 0, \\ 0, & if\ 0 < x \leq 1. \end{cases}$$

Clearly, $\delta = \{0_X, A, 1_X\}$ is fuzzy topology on X. By easy computations it follows that $B^{\frown} = B$ and $C^{\frown} = C$.Then $B \not{q} C^{\frown}$ and $C \not{q} B^{\frown}$. Hence B and C are fuzzy semi-preseparated. But $B_{-} = 1_X$ and $C_{-} = A'$, so that BqC_{-} and CqB_{-}. Thus B and C are not fuzzy semi-separated.

Again, $B = B \cup C$ and B, C are fuzzy semi-preseparated. It implies that B is not fuzzy semi-preconnected. We show that B is fuzzy semi-connected. In fact, let $B = D \cup E$, where D and E are non-null fuzzy sets in X. Then either $D(0) = 0.7$ or $E(0) = 0.7$. Suppose $D(0) = 0.7$, then $D_{-} = 1_X$. Clearly, EqD_{-}. Thus D and E cannot be fuzzy semi-separated. Hence B is fuzzy semi-connected.

Theorem 3.5. Let A and B be non-null fuzzy sets in an fts X.

(1) If A and B are fuzzy semi-preseparated, and C, D are non-null fuzzy sets such that $C \leq A, D \leq B$, then C and D are also fuzzy semi-preseparated.

(2) If $A \not{q} B$ and either both are fuzzy semi-preopen or both are fuzzy semi-preclosed, then A and B are fuzzy semi-preseparated.

(3) If A, B are either both fuzzy semi-preopen or both fuzzy semi-preclosed, then $A \cap B'$ and $B \cap A'$ are fuzzy semi-preseparated.

Proof. We prove only (3). Let A and B be both fuzzy semi-preopen. Since

$A \cap B' \leq B'$, $(A \cap B')^{\frown} \leq B'$

and hence $(A \cap B')^{\frown} \not{q} B$. Then

$(A \cap B')^{\frown} \not{q} (B \cap A')$.

Again since

$B \cap A' \leq A', (B \cap A')^{\frown} \leq A'$

and hence $(B \cap A')^{\frown} \not{q} A$. Then

$(B \cap A')^{\frown} \not{q} (A \cap B')$.

Thus $A \cap B'$ and $B \cap A'$ are fuzzy semi-preseparated.

Similarly, we can prove when A and B are fuzzy semi-preclosed.

Theorem 3.6. Two non-null fuzzy sets A and B are fuzzy semi-preseparated iff there exist two fuzzy semi-preopen sets U and V such that

$A \leq U, B \leq V, A \not{q} V$ and $B \not{q} U$.

Proof. For two fuzzy semi-preseparated sets A and B, $B \leq (A^{\frown})' = V$ (say) and $A \leq (B^{\frown})' = U$ (say), where V and U are clearly fuzzy semi-preopen, then $V \not{q} A^{\frown}$ and $U \not{q} B^{\frown}$. Thus $A \not{q} V$ and $B \not{q} U$.

Conversely, let U and V be fuzzy semi-preopen sets such that $A \leq U$, $B \leq V$, $A \not{q} V$ and $B \not{q} U$. Then $A \leq V'$, $B \leq U'$. Hence $A^{\frown} \leq V'$ and $B^{\frown} \leq U'$, which in turn imply that $A^{\frown} \not{q} B$ and $B^{\frown} \not{q} A$. Thus A and B are fuzzy semi-preseparated.

Theorem 3.7. Let A be a non-null fuzzy semi-preconnected set in X. If $A \leq B \leq A^{\frown}$, then B is also fuzzy semi-preconnected.

Proof. If B is not fuzzy semi-preconnected in X, then there exist fuzzy semi-preseparated sets C and D in X such that $B = C \cup D$. Let $E = A \cap C$ and $F = A \cap D$. Then $A = E \cup F$. Since $E \leq C$ and $F \leq D$, by Theorem 3.5(1), E and F are fuzzy semi-preseparated, contradicting the fuzzy semi-preconnectedness of A. Thus B is fuzzy semi-preconnected.

Theorem 3.8. Let A be a non-null fuzzy semi-preconncted set in X, and C and D be two fuzzy semi-preseparated sets in X. If $A \leq C \cup D$, then $A \leq C$ or $A \leq D$.

Proof. Suppose that $A \cap D \neq 0_X$ and $A \cap C \neq 0_X$. By Theorem 3.5(1), $A \cap C$ and $A \cap D$ become fuzzy semi-preseparated sets such that

$A = (A \cap C) \cup (A \cap D)$,

contradicting the fuzzy semi-preconnectedness of A. Hence $A \leq C$ or $A \leq D$.

Theorem 3.9. Let $\{A_i : i \in I\}$ be a collection of fuzzy semi-preconnected sets in the fts X. Suppose there exists a $j \in I$ such that A_i and A_j are fuzzy semi-preseparated for each $i \in I$. Then $A = \bigcup\{A_i : i \in I\}$ is fuzzy semi-preconnected.

Proof. If A is not fuzzy semi-preconnected, then $A = B \cup C$, where B and C are fuzzy semi-preseparated in X. Since A_i is fuzzy semi-preconnected for each $i \in I$, by Theorem 3.8, $A_i \leq B$ or $A_i \leq C$. Specifically, we have $A_j \leq B$ or $A_j \leq C$. We may assume that $A_j \leq B$. Then for each $i \neq j, A_i \leq B$. In fact, if $A_j \not\leq B$, then $A_i \leq C$. By Theorem 3.5(1) A_j and A_i are fuzzy semi-preseparated.

This is a contradiction. Hence for each $i \in I, A_i \leq B$. It follows that

$$A = \bigcup\{A_i : i \in I\} \leq B,$$

clearly, $C = 0_X$.Thus A is fuzzy semi-preconnected.

The following corollary is obvious.

Corollary 3.10. Let $\{A_i : i \in I\}$ be a collection of fuzzy semi-preconnected sets in X. If $\bigcap\{A_i : i \in I\} \neq 0_X$, then $\bigcup\{A_i : i \in I\}$ is fuzzy semi-preconnected.

Theorem 3.11. Let $f : X \to Y$ be a one-to-one fuzzy SP-irresolute mapping. If A is a fuzzy semi-preconnected set in X, then so is $f(A)$ in Y.

Proof. If possible, let $f(A)$ be not fuzzy semi-preconnected in Y. Then there exist fuzzy semi-preseparated sets B and C in Y such that

$$f(A) = B \cup C.$$

Since B and C are fuzzy semi-preseparated, by Theorem 3.6 there exist two fuzzy semi-preopen sets U and V such that

$$B \leq U, C \leq V, B \not{q} V \text{ and } C \not{q} U.$$

Now, f being fuzzy SP-irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are fuzzy semi-preopen sets in X, and

$$\begin{aligned} A &= f^{-1}f(A) \\ &= f^{-1}(B \cup C) \\ &= f^{-1}(B) \cup f^{-1}(C). \end{aligned}$$

For $B \not{q} V$ and $C \not{q} U$, we have $B \leq V'$ and $C \leq U'$, i.e.,

$$f^{-1}(B) \leq (f^{-1}(V))' \text{ and}$$
$$f^{-1}(C) \leq (f^{-1}(U))'.$$

Hence

$$f^{-1}(B) \not{q} f^{-1}(V) \text{ and}$$
$$f^{-1}(C) \not{q} f^{-1}(U).$$

By Theorem 3.6, $f^{-1}(B)$ and $f^{-1}(C)$ are fuzzy semi-preseparated in X. Thus we arrive at a contradiction.

Corollary 3.12. Let $f : X \to Y$ be a fuzzy SP-irresolate mapping. If X is fuzzy semi-perconnected, then so is $f(X)$.

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