A counter example

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Abstract: In this note, we disprove two results in [1] using a counter example.
Keywords: Fuzzy subgroup, fuzzy normal subgroup.

1. Preliminaries

Here G will denote a finite group.

Definition 1.1 A mapping A: $G\to[0, 1]$ is called a fuzzy subset of G.

Definition 1.2 A fuzzy subset A of G is called a fuzzy subgroup iff for every $x, y\in G$,

(1) $A(x, y) \geq \min\{ A(x), A(y)\}$;

(2) $A(x) = A(x^{-1})$;

(3) $A(e) = 1$.

Definition 1.3 A fuzzy subgroup A of G is called to be normal if

$$A(xyx^{-1}) \geq A(y),\ x, y\in G.$$

Definition 1.4 Let A be a fuzzy subset of G. Then the subset $G_A^t = \{x \in G : A(x) \geq t\}$, $t\in[0, 1]$ is called the t-levek subset of G under A.

Lemma 1.5 If A is a fuzzy subgroup of G, then every t-level subset G_A^t is a subgroup of G, $t\in\mathrm{Im}(A)$.

Lemma 1.6 If A is a fuzzy subset of G such that every t-level subset G_A^t of G is a subgroup of G,$\forall\ t\in\mathrm{Im}(A)$, then A is a fuzzy subgroup of G.

Let A be a fuzzy subgroup of G such that $\mathrm{Im}(A) = \{t_0, t_1, \ldots, t_n\}$, $t_0> t_1>\ldots>t_n$. Then there exist a chain of subgroups

$$G_A^{t_0} \subset G_A^{t_1} \subset \cdots \subset G_A^{t_n} = G \tag{1}$$

Otherwise, if

$$H_0\subset H_1\subset\ldots\subset H_n= G \tag{2}$$

is a chain of subgroups, then there exists a fuzzy subgroup B of G whose level subgroups are the elements of the chain (2).

Lemma 1.7 If A is a normal fuzzy subgroup of G, then the chain (1) is a normal chain and vice verse. If the chain (2) is normal, then B is a normal fuzzy subgroup.

Definition 1.8 If $x, y\in G$, then $x^{-1}y^{-1}xy$ is called the commutator of x, y and it is denoted by $[x, y]$.

Lemma 1.9 [2, Lemma 3.2] Let A be a fuzzy subgroup of G, let $x\in G$. Then

$$A(xy) = A(y),\ \forall\ y \in G \Leftrightarrow A(x) = A(e).$$

2. Counter example

Mishref [1] gave the following results:

(Theorem 3.1, [1]) Let A be a normal fuzzy subgroup of G. Then Im(A) contains at most two elements of [0, 1].

(Proposition 3.4, [1]) Let A be a fuzzy subgroup of G. Then A is normal iff $A([x, y]) = 1, \forall x, y \in G$.

Let G be the "four-group", that is, G is a group of order 8 given by $G=\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, where we have $a^4 = e$, $b^2 = a^2$, $b^{-1}ab = a^{-1} = a^3$, $ab \neq ba$, then $\{e\}$, $H_1 = \{e, a^2\}$, $H_2 = \{e, a, a^2, a^3\}$ are normal subgroups of G, and

$$\{e\} \subset H_1 \subset H_2 \subset G.$$

Also, for every $x \in G$, let

$$A(x) = \begin{cases} 1, & x = e \\ \frac{1}{2}, & x \in H_1 - \{e\} \\ \frac{1}{3}, & x \in H_2 - H_1 \\ \frac{1}{4}, & x \in G - H_2 \end{cases}$$

From Definition 1.2 and Lemma 1.7, A is a normal fuzzy subgroup of G. Consequently, we have

Result 2.1 $|\mathrm{Im}(A)| = 4$.

Result 2.2 For $a, b \in G$, $A([a, b]) \neq 1$.

Therefore, the Result 2.1 disprove Theorem 3.1 in [1], the Result 2.2 disprove Proposition 3.4 in [1].

In fact, we have following propositions.

Proposition 2.3 Let A be a normal fuzzy subgroup of G. If there exist n normal subgroups of G, then Im(A) contains at most n+2 elements of [0, 1].

The proof is straightforward.

Proposition 2.4 Let A be a fuzzy subgroup of G, and

$$A([x, y]) = 1, \forall x, y \in G.$$

Then A is normal.

Proof. Since

$$A([x, y]) = 1 = A(e), \forall x, y \in G,$$

we have

$$A([x, y]y^{-1}) = A(y^{-1}), \forall x, y \in G$$

from Lemma 1.9. Consequently,

$$A(x^{-1}y^{-1}xyy^{-1}) = A(y^{-1}), \forall x, y \in G$$

and

$$A(x^{-1}yx) = A(y), \forall x, y \in G.$$

Therefore, A is normal from Definition 1.3.

References

[1] M. Atif Mishref, Normal fuzzy subgroups and fuzzy normal series of finite groups, *Fuzzy Sets and Systems* 72(1995) 379~383.

[2] N. P. Makherjee and P. Bhattacharya, fuzzy normal subgroups and fuzzy cosets, *Inform. Sci.* 34(1986) 225~239.