

Some Results On BZMV^{dM} - algebras

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Abstract: A BZMV^{dM} - algebra is a system endowed with a commutative and associative binary operator \oplus and two unusual orthocomplementation: a Kleen orthocomplementation \neg and a Brouwerian one \sim , and every BZMV^{dM} - algebra is both an MV- algebra and a distributive de-Morgan BZ-lattice. Our main results is that the following properties hold, for any BZMV^{dM} -algebra $(A, \oplus, \neg, \sim, 0), \forall x, y \in A, (1) \sim \sim x = x$ iff $\sim x \otimes x = x$ iff $x \oplus x = x, (2) x \wedge y = 0$ iff $x \leq \neg y, (3) \sim(x \oplus y) = \sim x \otimes \sim y, (4) x \oplus \sim x = 1$.

Keywords: BZMV^{dM} -algebra; Kleen orthocomplementation; Brouwerian orthocomplementation.

In 1997, G. Cattaneo, R. Giuntini and R. Pilla presented the concept of BZMV^{dM} - algebras^[1]. The main properties of this system is studied, from [1], any BZMV^{dM} - algebras are the same as the Stonian MV- algebras and a first representation theorem is proved.

The aim of this paper is to make further investigation of the features of BZMV^{dM} - algebras.

1. Preliminaries

Definition 1.1^[1] A BZMV^{dM} - algebra is a system $\langle A, \oplus, \neg, \sim, 0 \rangle$ where A is a non-empty set of elements, 0 is a constant element of A , \neg and \sim are unary operations on A , \oplus is a binary operation on A , obeying the following axioms:

$$(MB_1) \quad (x \oplus y) \oplus z = (y \oplus z) \oplus x,$$

$$(MB_2) \quad x \oplus 0 = x,$$

$$(MB_3) \quad \neg(\neg x) = x,$$

$$(MB_4) \quad \neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x,$$

$$(MB_5) \quad \sim x \oplus \sim \sim x = \sim \sim x,$$

$$(MB_6) \quad x \oplus \sim \sim x = \sim \sim x,$$

$$(MB_7) \quad \sim \neg[\neg(x \oplus \neg y) \oplus \neg y] = \neg(\sim x \oplus \neg \sim y) \oplus \neg \sim y.$$

From [1], making use of BZMV^{dM} - algebra structure, we can define some new operations:

$$1 := \neg 0, \quad (1)$$

$$x \otimes y := \neg(\neg x \oplus \neg y), \quad (2)$$

$$x \vee y := (x \otimes \neg y) \oplus y = \neg(\neg x \oplus y) \oplus y, \quad (3)$$

$$x \wedge y := (x \oplus \neg y) \otimes y = \neg[\neg(x \oplus \neg y) \oplus \neg y]. \quad (4)$$

Theorem 1.2^[1] If A is a BZMV^{dM} - algebra, then the following results are true:

- (5) $x \oplus y = y \oplus x,$
- (6) $(x \oplus y) \oplus z = x \oplus (y \oplus z),$
- (7) $x \oplus 1 = 1,$
- (8) $x \oplus \neg x = 1,$
- (9) $\neg(x \oplus \sim x) \oplus \sim x = 1,$
- (10) $\neg x \oplus \sim x = 1,$
- (11) $x \wedge \sim x = x,$
- (12) $\sim(x \wedge y) = \sim x \vee \sim y,$
- (13) $\sim(x \vee y) = \sim x \wedge \sim y, (i.e., x \leq y \Rightarrow \sim y \leq \sim x,)$
- (14) $x \wedge \sim x = 0,$
- (15) $\sim \sim x = \sim x,$
- (16) $\sim x \oplus \sim x = \sim x,$
- (17) $\neg 0 = \sim 0.$

2. Some Results

Theorem 2.1 Let A be a BZMV^{dM} - algebra, then we have

$$(18) \quad \forall x, y \in A, x \wedge y = 0 \text{ iff } y \prec x$$

$$(19) \text{ for } x, \text{ and } x \oplus x = x, \text{ then } \forall y \in A, x \wedge y = 0 \text{ iff } x \leq \neg y$$

Proof. (18) Assume $x \wedge y = 0$, then $y \wedge \sim x = (y \wedge \sim x) \vee 0 = (y \wedge \sim x) \vee (y \wedge \sim y) = \neg(x \wedge y) = y \wedge \sim 0 = y.$

Conversely, suppose $y \leq \sim x$, then trivially

$$x \wedge y = x \wedge (y \wedge \sim x) = y \wedge (x \wedge \sim x) = y \wedge 0 = 0.$$

(19) Suppose $x = x \oplus x$, and, by (1) of proposition 2.2 of [1], $x \otimes y = x \wedge y$, hence, $x \leq \neg y$ iff $x \wedge y = 0$.

Theorem 2.2 Let A be a BZMV^{dM}-algebra, then $\forall x, y \in A$, we have

$$\sim(x \oplus y) = \sim x \otimes \sim y$$

Proof. By theorem 1.2, we have $(x \oplus y) \wedge \sim(x \oplus y) = 0$, and $\sim(x \oplus y) \leq \sim x \otimes \sim y$.

By $x \oplus y = \sim \sim x \oplus \sim \sim y$, hence $\sim(\sim x \oplus \sim y) = \sim(x \oplus y)$. That is

$$\sim(\sim x \oplus \sim y) = (\sim x \oplus \sim y) = \neg(\neg \sim x \oplus \neg \sim y) = \sim x \otimes \sim y.$$

Theorem 2.3 Let A be a BZMV^{dM}-algebra, then $\forall x \in A$, we have

$$x \oplus \sim x = 1$$

Proof. Since $\sim(x \oplus \sim x) = \sim x \otimes \sim \sim x = \neg(\neg \sim x \oplus \neg \sim \sim x) = \neg(\neg \sim x \oplus \neg \neg \sim x)$.

By (8) of theorem 1.2, we have $\neg(\neg \sim x \oplus \neg \neg \sim x) = \neg 1 = 0$. So, $x \oplus \sim x = 1$.

Definition 2.4 Let A be a BZMV^{dM}-algebra, then $\forall x, y \in A$, we introduce the following new operation " \rightarrow ":

$$x \rightarrow y = \neg x \oplus y$$

Theorem 2.5 Let A be a BZMV^{dM}-algebra, then $\forall x, y, z \in A$, the following holds:

$$(20) \quad x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z),$$

$$(21) \quad x \rightarrow y = \neg y \rightarrow \neg x,$$

$$(22) \quad (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$$

$$(23) \quad x \rightarrow x = 1.$$

Proof. (20) $x \rightarrow (y \rightarrow z) = \neg x \oplus (y \rightarrow z) = \neg x \oplus (\neg y \oplus z) = (\neg x \oplus \neg y) \oplus z$

$$= (\neg y \oplus \neg x) \oplus z = \neg y \oplus (\neg x \oplus z) = \neg y \oplus (x \rightarrow z)$$

$$= y \rightarrow (x \rightarrow z).$$

$$(21) \quad x \rightarrow y = \neg x \oplus y = y \oplus \neg x = \neg(\neg y) \oplus \neg x = \neg y \rightarrow \neg x.$$

$$(22) \quad (x \rightarrow y) \rightarrow y = \neg(x \rightarrow y) \oplus y = \neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$$

$$= \neg(\neg y \oplus x) \oplus x = (\neg y \oplus x) \rightarrow x = (y \rightarrow x) \rightarrow x.$$

$$(23) \quad x \rightarrow x = \neg x \oplus x = 1$$

In any BZMV^{dM}-algebra the elements which are idempotent with respect to the operation \oplus ,

the condition leads to the following:

Theorem 2.6 Let A be a $BZMV^{dM}$ -algebra, then $\forall x, y, z \in A$, the following holds:

$$(24) \quad x \rightarrow (y \oplus z) = (x \rightarrow y) \oplus (x \rightarrow z),$$

$$(25) \quad \neg x \rightarrow x = x.$$

Proof. (24) Since $(x \rightarrow y) \oplus (x \rightarrow z) = (\neg x \oplus y) \oplus (\neg x \oplus z)$

$$= (\neg x \oplus \neg x) \oplus (y \oplus z) = \neg x \oplus (y \oplus z) = x \rightarrow (y \oplus z).$$

$$(25) \quad \neg x \rightarrow x = \neg \neg x \oplus x = x \oplus x = x.$$

Let us introduce the following order relation " \leq ":

$$x \leq y \text{ iff } x \rightarrow y = 1 \text{ iff } \neg x \oplus y = 1$$

Theorem 2.7 Let A be a $BZMV^{dM}$ -algebra, then $\forall x, y, z \in A$, the following holds:

$$(i) \text{ if } x \leq y, \text{ then } y \rightarrow z \leq x \rightarrow z,$$

$$(ii) \text{ if } x \leq y, \text{ then } \neg y \oplus z \leq \neg x \oplus z.$$

Proof. (i) Since $(y \rightarrow z) \rightarrow (x \rightarrow z) = x \rightarrow ((y \rightarrow z) \rightarrow z) = x \rightarrow ((z \rightarrow y) \rightarrow y)$

$$= (z \rightarrow y) \rightarrow (x \rightarrow y) = (z \rightarrow y) \rightarrow 1 = \neg(z \rightarrow y) \oplus \neg 0 = \neg 0 = 1,$$

so, $y \rightarrow z \leq x \rightarrow z$.

$$(ii) \text{ Since } (\neg y \oplus z) \rightarrow (\neg x \oplus z) = (y \rightarrow z) \rightarrow (x \rightarrow z) = x \rightarrow ((y \rightarrow z) \rightarrow z) = 1,$$

hence $\neg y \oplus z \leq \neg x \oplus z$.

References

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