Some Properties of Complete Lattice

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Abstract: In this paper, properties of the subset of complete lattice, complemented lattice, complete complemented lattice had been discussed.
Keyword: complete lattice, complemented lattice.

1.Preliminares

Definition 1.1: $(L, \wedge, \vee, ')$ is called a complemented lattice if

(1) $(L, \wedge, \vee)$ is a lattice;

(2) $':L\to L$ is a inverted sequence convolution mapping, i.e. for any $a, b \in L$

① $a\le b \Rightarrow b'\le a'$; ② $(a')'=a$.

Definition 1.2: $(L, \wedge, \vee)$ is called a complete lattice if

(1) $(L, \wedge, \vee)$ is a lattice;

(2) for any subset A of L, supA and infA exist.

Theorem 1.1 Let L be a complemented lattice with universal bound, then in L, generalized De Morgan's rules hold:

$$(\bigvee_{\alpha\in I} a_\alpha)' = \bigwedge_{\alpha\in I} a_\alpha{}' \qquad (\bigwedge_{\alpha\in I} a_\alpha)' = \bigvee_{\alpha\in I} a_\alpha{}'$$

2. Properties

Definition 2.1: Let L be a complete lattice. $E \subseteq L$, $E\neq\phi$. E is called a zone of L if the following conditions are satisfied.

(1) $\forall a_\alpha \in E$ $(\alpha\in I)$, we have $\bigvee_{\alpha\in I} a_\alpha \in E$;

(2) $a\le b$ and $b\in E \Rightarrow a\in E$.

Dually, E is called a dual zone of L if the following conditions are satisfied.

(1) $\forall a_\alpha \in E$ $(\alpha\in I)$, we have $\bigwedge_{\alpha\in I} a_\alpha \in E$;

(2) $a\ge b$ and $b\in E \Rightarrow a\in E$.

Obviously, if E is a zone (dual zone) of L, then

(1) $O\in E$ (or $I\in E$); (2) E is a closed sublattice of L.

According to Definition 2.1, we regard the concept of zone (deal zone) as the generalization of the concept of ideal (dual ideal).

Theorem 2.1 Let L be a complete lattice. E be a non-empty subset of L. Then

(1) E is a zone (dual zone) of L if and only if E satisfied the condition:

$$\bigvee_{\alpha\in I} a_\alpha \in E \ (\bigwedge_{\alpha\in I} a_\alpha \in E) \Leftrightarrow \forall \alpha\in I,\ a_\alpha \in E.$$

E is a zone (dual zone) of L if and only if E is a closed sublattice of L, and E satisfies the condition:

(2) $\forall\ a_\alpha \in E,\ b_\beta \in L(\alpha\in I, \beta\in T) \Rightarrow (\bigwedge_{\alpha\in I} a_\alpha)\wedge(\bigwedge_{\beta\in T} b_\beta) \in E$ $((\bigwedge_{\alpha\in I} a_\alpha)\vee(\bigwedge_{\beta\in T} b_\beta)\in E)$.

Theorem 2.2 Let L be a complete complemented lattice, E⊆L and $\forall a\in$L, There is only one for a and a' belongs to E, then

(1) E is a closed sublattice of L if and only if L–E is a closed sublattice of L;

(2) E is a zone of L if and only if L–E is a dual zone of L;

(3) E is a dual zone if and only if E satisfies the condition: $\bigvee_{\alpha\in I} a_\alpha \in E \Leftrightarrow \exists\beta\in I,\ a_\beta\in E.$ (1)

Proof:

(1) Clearly, E is a sublattice of L if and only if L–E is a sublattice of L. Suppose E be a closed sublattice of L, we consider any nonempty subset T of L-E. Since T⊆L–E, so $\forall a\in T$, we have $a'\in E$. Thus, by Theorem 1.1

$$\bigvee_{\alpha\in I} a_\alpha = (\bigwedge_{\alpha\in I} a_\alpha')' \in L-E \qquad \bigwedge_{\alpha\in I} a_\alpha = (\bigvee_{\alpha\in I} a_\alpha')' \in L-E$$

Therefore, L–E is a closed sublattice. Sufficiency is proved as above.

(2) Let E be a zone of L. If $a\geq b$ and $b\in L-E$, the $a'\leq b'$ and $b'\in E$.
Since E is a zone of L, we have $a'\in E$, i.e. $a\in L-E$. Also, $\forall a_\alpha\in L-E$ ($\alpha\in I$), we have $a'_\alpha\in E$. So $\bigvee_{\alpha\in I} a_\alpha' \in E$. By Theorem 1.1, $\bigwedge_{\alpha\in I} a_\alpha = (\bigvee_{\alpha\in I} a_\alpha')' \in L-E$. Therefore, L–E is a dual zone of L.

(3) Suppose E, If $a\in I$, $a_\alpha\notin E$, then $a_\alpha'\in E$ ($\forall\alpha\in I$).

Since E is a dual zone of L, $\bigvee_{\alpha\in I} a_\alpha' \in E$, Thus $\bigwedge_{\alpha\in I} a_\alpha = (\bigvee_{\alpha\in I} a_\alpha')' \notin E$. This is a contradiction. So there exists $\beta\in I$, such that $a_\beta\in E$. If $\bigvee_{\alpha\in I} a_\alpha \notin E$ $\bigwedge_{\alpha\in I} a_\alpha' = (\bigvee_{\alpha\in I} a_\alpha)' \in E$.

Since E is a dual zone of L, by Theorem 2.1(1): $\forall a\in I$, $a_\alpha'\in E$, i.e. $a_\alpha\notin E$, This is a contradiction with "$\exists\beta\in I$, such that $a_\beta\in E$ ", thus $\bigvee_{\alpha\in I} a_\alpha \in E$.

To sum up, when E is a dual zone of L, (1) holds.

Necessity:

If $\exists\beta\in I$, such that $a_\beta\notin E$, i.e. $a_\beta'\in E$. By (1) we have $\bigvee_{\alpha\in I} a_\alpha' \in E$, that is $\bigwedge_{\alpha\in I} a_\alpha = (\bigvee_{\alpha\in I} a_\alpha')' \notin E$. This contradict with $\bigwedge_{\alpha\in I} a_\alpha \in E$. So $\bigwedge_{\alpha\in I} a_\alpha \in E \Rightarrow \forall\alpha\in I,\ a_\alpha\in E.$ (2)

Conversely, suppose $\forall\alpha\in I$, $a_\alpha\in E$, if $\bigwedge_{\alpha\in I} a_\alpha \notin E$, then $\bigvee_{\alpha\in I} a_\alpha' = (\bigwedge_{\alpha\in I} a_\alpha)' \in E$. By (1), $\exists\beta\in I$, such that $a_\beta'\in E$, i.e. $a_\beta\notin E$. This is a contradiction with "$\forall\alpha\in I$, $a_\alpha\in E$ ". Thus

$$\forall\alpha\in I,\ a_\alpha\in E \Rightarrow \bigwedge_{\alpha\in I} a_\alpha \in E \qquad (3)$$

By (2), (3) and Theorem 2.1(1), we have the result that E is a dual zone of L.

Corollary 2.1 Let L be a complemented lattice, E⊆L and $\forall\alpha\in I$, there is only one for a and a' belongs to E. Then: (1) E is a sublattice of L if and only if L–E is a sublattice of L.

(2)E is an ideal of L if and only if L–E is a dual ideal of L.

(3) E is a dual ideal if and only if E satisfies the condition: $a\vee b\in E \Leftrightarrow a\in E$ or $b\in E$.

References

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