Fuzzy Algebras and Fuzzy Quotient Algebras over Fuzzy Fields

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Abstract: The concepts of fuzzy algebra and fuzzy ideal over a fuzzy field are introduced and some properties of fuzzy algebra and fuzzy ideal are discussed. At last, several isomorphism theorems for fuzzy quotient algebras are established.

Keywords: Fuzzy field; fuzzy algebra; fuzzy ideal; fuzzy quotient algebra

1. Fuzzy field and fuzzy algebra

In this paper, L always stands for a complete distributive lattice with the smallest element 0 and the largest element 1. By a fuzzy subset of a nonempty set we mean a mapping from the nonempty set to L. Unless specially statement, X denotes a field and Y denotes an algebra over X.

The concept of fuzzy algebra over a fuzzy field was defined first by S.Nanda[1] and was redefined by Gu and Lu[2] . Now we give the following definitions in order to define reasonably fuzzy quotient algebra.

Definition1.1 Let F be a fuzzy subset of a field X. If for all $\lambda_1, \lambda_2 \in X$,

(i) $F(\lambda_1 - \lambda_2) \geq F(\lambda_1) \wedge F(\lambda_2)$,

(ii) $F(\lambda_1 \lambda_2^{-1}) \geq F(\lambda_1) \wedge F(\lambda_2), \lambda_2 \neq 0$,

then F is called a fuzzy field of X.

Clearly, If F is called a fuzzy field of X , then for all $\lambda \neq 0$,we have $F(0) \geq F(1) \geq F(\lambda)$.

Definition1.2 Let F be a fuzzy field of X and Y be an algebra over X and A be a fuzzy subset of Y. If for all $y_1, y_2 \in Y$ *and* $\lambda \in X$,

(i) $A(y_1 - y_2) \geq A(y_1) \wedge A(y_2)$,

(ii) $A(\lambda y_1) \geq F(\lambda) \wedge A(y_1)$,

(iii) $A(y_1 y_2) \geq A(y_1) \wedge A(y_2)$,

then A is called a fuzzy algebra of Y over fuzzy field F. In brief, we call A a fuzzy F_X^Y – algebra.

It is clear that $A(0) \geq A(y)$ for all $y \in Y$ if A is a fuzzy F_X^Y – algebra.

Definition 1.3 Let A be a fuzzy F_X^Y – algebra. If for all $y_1, y_2 \in Y$ *and* $\lambda \in X$,

(i) $A(y_1 y_2) \geq A(y_1) \vee A(y_2)$,

(ii) $A(\lambda y_1) \geq F(\lambda) \vee A(y_1)$,

Then A is called a fuzzy F_X^Y – ideal.

It is clear that $A(0) \geq F(0)$ if A is a fuzzy F_X^Y – ideal.

2. Fuzzy quotient algebra

In the following statement, Y always stands for an algebra over a field X and F stands for a fuzzy field of X.

Definition 2.1 Let A be a fuzzy F_X^Y – ideal, then for all $y \in Y$ we define the fuzzy subset $y + A$ as follows: $(y + A)(y_1) = A(y_1 - y), \forall y_1 \in Y$.

Proposition 2.2 Let A be a fuzzy F_X^Y – ideal, then for all $y_1, y_2 \in Y$, we have

$$y_1 + A = y_2 + A \Leftrightarrow A(y_1 - y_2) = A(0).$$

Proof. Necessity: $y_1 + A = y_2 + A \Rightarrow A(y_1 - y_2) = (y_2 + A)(y_1) = (y_1 + A)(y_1) = A(0)$.

Sufficiency: $\forall y \in Y$, we have $(y_1 + A)(y) = A(y - y_1) = A((y - y_2) - (y_1 - y_2))$

$$\geq A(y - y_2) \wedge A(y_1 - y_2) = A(y - y_2) = (y_2 + A)(y)$$

That is, $y_1 + A \geq y_2 + A$.

Similarly, $y_2 + A \geq y_1 + A$. So, $y_1 + A = y_2 + A$.

Proposition 2.3 Let A be a fuzzy F_X^Y – ideal, then for all $y_1, y_2, x_1, x_2 \in Y, \lambda \in X$, we have

(i) $x_1 + A = y_1 + A, x_2 + A = y_2 + A \quad \Rightarrow (x_1 + x_2) + A = (y_1 + y_2) + A, \; x_1 x_2 + A = y_1 y_2 + A$,

(ii) $x_1 + A = y_1 + A \Rightarrow \lambda x_1 + A = \lambda y_1 + A$.

Proof.(i) $A((x_1 + x_2) - (y_1 + y_2)) \geq A(x_1 - y_1) \wedge A(x_2 - y_2) = A(0)$,

$$A(x_1 x_2 - y_1 y_2) = A((x_1 - y_1)x_2 + y_1(x_2 - y_2)) \geq A(x_1 - y_1) \wedge A(x_2 - y_2) = A(0),$$

So $A((x_1 + x_2) - (y_1 + y_2)) = A(x_1 x_2 - y_1 y_2) = A(0)$.

From Proposition 2.2 we have $(x_1 + x_2) + A = (y_1 + y_2) + A \; and \; x_1 x_2 + A = y_1 y_2 + A$.

(ii) $A(\lambda x_1 - \lambda y_1) = A(\lambda(x_1 - y_1)) \geq A(x_1 - y_1) = A(0)$, i.e. $A(\lambda x_1 - \lambda y_1) = A(0)$.

Hence, $\lambda x_1 + A = \lambda y_1 + A$.

Proposition 2.4 Let A be a fuzzy F_X^Y – ideal, then Y/A is an algebra over X and $Y/A \cong Y/A_0$, where $Y/A = \{y + A \mid y \in Y\}$, $A_0 = \{y \in y \mid A(y) = A(0)\}$,

$$\begin{cases} (y_1 + A) + (y_2 + A) = (y_1 + y_2) + A, \\ (y_1 + A)(y_2 + A) = y_1 y_2 + A, & \forall y_1, y_2 \in Y, \forall \lambda \in X. \\ \lambda(y_1 + A) = \lambda y_1 + A, \end{cases}$$

Proof. From Proposition 2.3 we know that Y/A is an algebra over X and $f : Y/A \to Y/A_0$ defined by $f(y + A) = y + A_0$ is an isomorphism.

Hence, $Y/A \cong Y/A_0$.

Definition 2.5 Let A be a fuzzy F_X^Y – ideal, then Y/A is called fuzzy quotient algebra of Y concern with A.

Proposition 2.6 Let A be a fuzzy F_X^Y – ideal, then the fuzzy subset A/A_0 of Y/A_0 is a fuzzy algebra over the fuzzy field F, where $A/A_0\,(y+A_0)=A(y),\ \forall y\in Y.$

Proof. It is clear that A/A_0 is well defined. Let $y_1, y_2\in Y, \lambda\in X$, then we have

$$A/A_0\,((y_1+A_0)-(y_2+A_0)) = A/A_0\,((y_1-y_2)+A_0) = A(y_1-y_2)\ge A(y_1)\wedge A(y_2)$$
$$= A/A_0\,(y_1+A_0)\wedge A/A_0\,(y_2+A_0),$$

Similarly $A/A_0\,((y_1+A_0)(y_2+A_0))\ge A/A_0\,(y_1+A_0)\wedge A/A_0\,(y_2+A_0)$,

$$A/A_0\,(\lambda(y_1+A_0)) = A/A_0\,(\lambda y_1+A_0) = A(\lambda y_1)\ge F(\lambda)\wedge A(y_1) = F(\lambda)\wedge A/A_0\,(y_1+A_0).$$

Hence A/A_0 is a fuzzy algebra of Y/A_0 over the fuzzy field F.

Proposition 2.7 Let A be a fuzzy F_X^Y – ideal, then the fuzzy subset A^* of Y/A is a fuzzy algebra over the fuzzy field F, where $A^*(y+A)=A(y),\ \forall y\in Y.$

Proof. The proof is similar to that of Proposition 2.6 and hence omitted.

References

[1] S.Nanda, Fuzzy algebras over a fuzzy field, Fuzzy Sets and Systems,37(1990):99-103

[2] Gu Wenxiang and Lu Tu, Fuzzy algebras over fuzzy fields redefined, Fuzz Sets and Systems,53(1993):105-107

[3] Dang Faning, Fuzz algebra and fuzzy quotient algebra, Fuzzy Systems and Mathematics, 4(19960):68-75(in Chinese)

[4] Liu Wangjin, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems, 8(1982):133-139