Grey Congruence relations of Groups*

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Abstract: In this paper, we introduce the concepts of Congruence relation of Groups and give the important properties of it.

Keywords: Grey Set,Group, Grey Congruence relation.

Introduction

In[1],some authors studied the theorem of grey subset and grey subgroups, in[2-3],some authors studied fuzzy relations on rings and groups.In this paper,we study grey congruence relations of groups still further based on [1-3],First, we give the Definition of grey relations.

Let G be any set and L a bounded lattice with 1 and 0,then a grey relation A in G is characterized by two mapping: $\overline{U}_A : G\times G \to L$, $\underline{U}_A : G\times G \to L$ where $\overline{U}_A \geq \underline{U}_A$. The relation $A_{[\lambda_1,\lambda_2]}=\{x|x\in G,\ \overline{U}_A(x)\geq\lambda_2, \underline{U}_A\geq\lambda_1\}$ is called a $[\lambda_1, \lambda_2]$—level relation of grey relation A.

Grey Congruence relations of Group

Definition 1. The grey relation A of G is called a grey congruence relation if we have

(1) For any $x\in G$, $\overline{U}_A(x,x)=\underline{U}_A(x,x)=1$;

(2) For any $x,y\in G$, $\overline{U}_A(x,y)=\overline{U}_A(y,x), \underline{U}_A(x,y)=\underline{U}_A(y,x)$;

(3) $\overline{U}_A\bullet\overline{U}_A\subseteq\overline{U}_A, \underline{U}_A\bullet\underline{U}_A\subseteq\underline{U}_A$;

(4) For any $x,y,z\in G$, $\overline{U}_A(xz,yz)\geq\overline{U}_A(x,y), \overline{U}_A(zx,zy)\geq\overline{U}_A(x,y)$,
$\underline{U}_A(xz,yz)\geq\underline{U}_A(x,y), \underline{U}_A(zx,zy)\geq\underline{U}_A(x,y)$.

A is called a grey equivalence relation if A satisfy (1)、(2)and (3) of Definition 1.

Theorem 1. Let A be grey relation of G,then A is a grey congruence relation of G if for any $[\lambda_1\lambda_2],\in L$, $A_{[\lambda_1,\lambda_2]}$ be congruence relation of G.

Proof: Necessity: Because A is a grey congruence relation of G, then $A_{[\lambda_1,\lambda_2]}$ is easily established to be equivalence relation by (1)、(2)and(3) of Definition 1. For any $x,y\in G$, if $x\equiv y\,(A_{[\lambda_1,\lambda_2]})$,then $(x,y)\in A_{[\lambda_1,\lambda_2]}$, so for any $z\in G$, By (4) of Definition 1,we have

$\overline{U}_A(xz,yz)\geq\overline{U}_A(x,y)\geq\lambda_2$,

$\overline{U}_A(zx,zy)\geq\overline{U}_A(x,y)\geq\lambda_2$,

$\underline{U}_A(xz,yz)\geq\underline{U}_A(x,y)\geq\lambda_1$,

$\underline{U}_A(zx,zy)\geq\underline{U}_A(x,y)\geq\lambda_1$,

That is $(xz,yz),(zx,zy)\in A_{[\lambda_1,\lambda_2]}$,and $xz\equiv yz\,(A_{[\lambda_1,\lambda_2]}), zx\equiv zy\,(A_{[\lambda_1,\lambda_2]})$,

So $A_{[\lambda_1,\lambda_2]}$ is congruence relation of G.

Sufficiency: For any $[\lambda_1, \lambda_2]\in L$, $A_{[\lambda_1,\lambda_2]}$ is congruence relation of G,then A is easily

* It is imbursed by Shan Dong Natural Science Fund.

established to be grey equivalence relation. Because $A_{[\lambda_1,\lambda_2]}$ is congruence relation of G, for any $x, y, z \in G$, we have,

$$\overline{U}_{A_{\lambda_2}}(xz, yz) \geq \overline{U}_{A_{\lambda_2}}(x, y),$$

$$\overline{U}_{A_{\lambda_2}}(zx, zy) \geq \overline{U}_{A_{\lambda_2}}(x, y),$$

there into $\overline{U}_{A_{\lambda_2}}(x, y) = \begin{cases} 1, & if \quad (x,y) \in A_{\lambda_2} \\ 0, & if \quad (x,y) \notin A_{\lambda_2} \end{cases}$

So we have $\overline{U}_A(xz, yz) = \sup\limits_{\lambda_2 \in L} \lambda_2 \overline{U}_{A_{\lambda_2}}(xz, yz) \geq \sup\limits_{\lambda_2 \in L} \lambda_2 \overline{U}_{A_{\lambda_2}}(x, y) = \overline{U}_A(x, y)$,

Similarly, we have $\overline{U}_A(zx, zy) \geq \overline{U}_A(x, y)$,

$$\underline{U}_A(xz, yz) \geq \underline{U}_A(x, y),$$

$$\underline{U}_A(zx, zy) \geq \underline{U}_A(x, y).$$

Then A is a grey congruence relation of G.

Definition 2. Let G be a group, A and B be grey relations of G, define grey relation of $G: A \cap B$, , $A \bullet B$, by:

(1) $\overline{U}_{A \cap B}(x,y) = \min\{\overline{U}_A(x,z), \overline{U}_B(z,x)\}$, $\underline{U}_{A \cap B}(x,y) = \min\{\underline{U}_A(x,z), \underline{U}_B(z,x)\}$;

(2) $\overline{U}_{A \bullet B}(x,y) = \sup\limits_{z \in G}(\min(\overline{U}_A(x,z), \overline{U}_B(z,x))$,

$\underline{U}_{A \bullet B}(x,y) = \sup\limits_{z \in G}(\min(\underline{U}_A(x,z), \underline{U}_B(z,x))$;

respectively.

Theorem 2. Let A and B be grey congruence relations of G, then $A \bullet B = B \bullet A$.

Proof: For any $x, y \in G$,

$$\overline{U}_{A \bullet B}(x, y) = \sup_{z \in G}(\min(\overline{U}_A(x, z), \overline{U}_B(z, y)))$$

$$\leq \sup_{z \in G}(\min(\overline{U}_A(yz^{-1}x, yz^{-1}z), \overline{U}_B(zz^{-1}x, yz^{-1}x)))$$

$$= \sup_{z \in G}(\min(\overline{U}_B(x, yz^{-1}x), \overline{U}_A(yz^{-1}x, y))) = \overline{U}_{B \bullet A}(x, y)$$

That is $\overline{U}_{A \bullet B} \subseteq \overline{U}_{B \bullet A}$.

Similarly, we have

$\overline{U}_{B \bullet A} \subseteq \overline{U}_{A \bullet B}$, that is $\overline{U}_{A \bullet B} = \overline{U}_{B \bullet A}$.

Similarly, we have

$\underline{U}_{A \bullet B} = \underline{U}_{B \bullet A}$, then $A \bullet B = B \bullet A$.

Theorem 3 Let A and B be grey congruence relations of G, then $A \bullet B$ is a grey congruence relation of G.

Proof: (1) Because A and B be grey congruence relations of G, then for any $x, y \in G$, we have

$$\overline{U}_{A \bullet B}(x, x) = \sup_{z \in G}(\min(\overline{U}_A(x, z), \overline{U}_B(z, x)) \geq \min(\overline{U}_A(x, x), \overline{U}_B(x, x)) = 1,$$

$$\underline{U}_{A \bullet B}(x, x) = \sup_{z \in G}(\min(\underline{U}_A(x, z), \underline{U}_B(z, x)) \geq \min(\underline{U}_A(x, x), \underline{U}_B(x, x)) = 1,$$

then $\overline{U}_{A \bullet B}(x, x) = \underline{U}_{A \bullet B}(x, x) = 1$

(2) For any $x, y \in G$, we have

$$\overline{U}_{A \bullet B}(x, y) = \sup_{z \in G}(\min(\overline{U}_A(x, z), \overline{U}_B(z, y))) = \sup_{z \in G}(\min(\overline{U}_B(z, y), \overline{U}_A(x, z)))$$

$$= \overline{U}_{B \bullet A}(y, x) = \overline{U}_{A \bullet B}(y, x).$$

Similarly,we have $\underline{U}_{A\bullet B}\ (x,y)=\underline{U}_{A\bullet B}(y,x)$.

(3) That is distinctness, $(\overline{U}_{A\bullet B})\bullet(\overline{U}_{A\bullet B})\subseteq(\overline{U}_{A\bullet B})$, $(\underline{U}_{A\bullet B})\bullet(\underline{U}_{A\bullet B})\subseteq(\underline{U}_{A\bullet B})$.

(4) $\overline{U}_{A\bullet B}\ (xz,yz)=\sup\limits_{g\in G}\ (\min(\overline{U}_A\ (xz,g),\ \overline{U}_B\ (g,yz)))$

$$=\sup_{g\in G}\ (\min(\overline{U}_A\ (xz,gz),\ \overline{U}_B\ (gz,yz)))$$

$$\geq\sup_{g\in G}\ (\min(\overline{U}_A\ (x,g),\ \overline{U}_B\ (g,y)))=\overline{U}_{A\bullet B}(x,y).$$

Similarly,we have $\overline{U}_{A\bullet B}\ (zx,zy)\geq\overline{U}_{A\bullet B}\ (x,y)$, $\underline{U}_{A\bullet B}\ (xz,yz)\geq\underline{U}_{A\bullet B}\ (x,y)$, $\underline{U}_{A\bullet B}\ (zx,zy)\geq\underline{U}_{A\bullet B}\ (x,y)$.

Then $A\bullet B$ is a grey congrence relation of G.

Theorem 4 Let A and B be grey congruence relations of G, then $A\cap B$ is a grey congruence relation of G.

Theorem 5 Let L be a complete lattice, then $\bigcap\limits_{k\in I} A^{(k)}$ is a grey congrence relation of G, here $\{A^{(k)}\}_{k\in I}$ is a family of grey congrence relations of G. If I is a finite set,then for all $[\lambda_1,\lambda_2]\subseteq L$, $(\bigcap\limits_{k\in I} A^{(k)})_{[\lambda_1,\lambda_2]}=\bigcap\limits_{k\in I} A^{(k)}_{[\lambda_1,\lambda_2]}$.

Theorem 4 and Theorem 5 can be easily drawn.

Definition 3. Let A be a grey relation of G, for any $a\in G$, define grey relation $[a]A$ by two mapping: $[a]\overline{U}_A:G\to L,[a]\underline{U}_A:G\to L$. For any $x\in G$,the set $[a]A(x)=A(a,x)$.

Theorem 6 Let A be a grey equivalent relation of G, then for $x,y\in G$, $[x]A=[y]A$ iff $A(x,y)=1$.

Proof: Necessity: $\overline{U}_A(x,y)=[x]\overline{U}_A(y)=[y]\overline{U}_A(y)=\overline{U}_A(y,y)=1$,
similarly,we have $\underline{U}_A(x,y)=[x]\underline{U}_A(y)=[y]\underline{U}_A(y)=\underline{U}_A(y,y)=1$.Then $A(x,y)=1$.

Sufficiency:For any $x,y,z\in G$,

$[x]\overline{U}_A[z]=\overline{U}_A(x,z)\geq\min\{\overline{U}_A(x,y),\overline{U}_A(y,z)\}=\min\{1,\overline{U}_A(y,z)\}=\overline{U}_A(y,z)=[y]\overline{U}_A(z)$,so $[x]\overline{U}_A\supseteq[y]\overline{U}_A$.Similarly, we have $[y]\overline{U}_A\supseteq[x]\overline{U}_A$. Then $[x]\overline{U}_A=[y]\overline{U}_A$.Similarly, we have $[x]\underline{U}_A=[y]\underline{U}_A$.Then $[x]A=[y]A$.

Theorem 7 Let A is a grey congrence relation of G, then for any $[\lambda_1\lambda_2],\in L$, $([a]A)_{[\lambda_1\lambda_2]}=[a]A_{[\lambda_1,\lambda_2]}=\{x\in G\mid x\equiv a(A_{[\lambda_1,\lambda_2]})\}$.

Proof: For any $x\in[a]\overline{U}_{A\lambda_2}$, then $x\equiv a(\overline{U}_{A\lambda_2})$,that is $\overline{U}_{A\lambda_2}(a,x)=1$,then $\overline{U}_A(a,x)\geq\lambda_2$, $[a]\overline{U}_A(x)\geq\lambda_2$, so $x\in([a]\overline{U}_A)_{\lambda_2}$. Contrary for any $x\in([a]\overline{U}_A)_{\lambda_2}$,then $[a]\overline{U}_A(x)\geq\lambda_2$,that is $\overline{U}_A(a,x)\geq\lambda_2$,so $\overline{U}_{A\lambda_2}(a,x)=1$,and we have $x\in[a]\overline{U}_{A\lambda_2}$.Therefore $[a]\overline{U}_{A\lambda_2}=([a]\overline{U}_A)_{\lambda_2}$. Similarly,we have $[a]\underline{U}_{A\lambda_1}=([a]\underline{U}_A)_{\lambda_1}$. Therefore $([a]A)_{[\lambda_1,\lambda_2]})=[a]A_{[\lambda_1,\lambda_2]}$.

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