Group Action and Uniform HX-groups(1)

Zhang Chengyi

Dep. of Math. Hainan Teachers College, Haikou 571158, China

Dang Pingan

Dep. of Math. Zhumadian Teacher's College, Henan 463000, China

Zheng Qingan

Nanyang Teachers' College, Henan 473061,China

Abstract In this paper, the operational properties of the uniform HX-group are discussed. By the action of the group G* on the uniform HX-group, the conditions on a subset of the powerset P(G) to be an uniform HX-group is studied.

Keyword HX- group,Uniform HX-group,Group action,Upgrade .

The theoretical needs of the set-value mappings lead the birth of some mathematical structures. Prof. Li Hongxing etc[1,2,3] first introduced the concept of HX-group which originated the study of HX-group, moreover, some useful results are obtained. Since the operations in a HX-group is based on the operations of some elements in the base algebra ,it is worth to study how to represent directly these operations and to judge whether a subset of the powerset P(G) are a certain algebric structure. In this paper, the operational properties of the uniform HX-group are discussed. By the action of the group G* on the uniform HX-group, the conditions of a nonempty subset of the powerset P(G) to form an uniform HX-group is studied.

1. Introduction

Let G be an arbitrary group and P(G), the powerset of G.Under the subset multiplication

$$AB=\{ab \mid a \in A, b\in B\},$$

$P_0(G)=P(G)-\{\varnothing\}$ forms a semigroup which have the identity. A subgroup g of $P_0(G)$ is called a HX-group on G, and G, the generating group of g. The identity of g is denoted by E.

Let g be a HX-group on G. g is called a regular HX-group if $e\in E$ (e is the identity of G).Suppose $A\in g$, A^{-1} is the inverse element of A and $A^{(-1)}=\{x^{-1} \mid x\in A\}$ is called the inverse set of A. g is called an uniform HX-group if for all $A\in g$, $A^{(-1)}= A^{-1}$.Let $G^*=\cup\{A \mid A\in g\}$, G* is called the base elements set.Let g be a HX-group on G, we have following conclusions:

Lemma 1.1[2] For any $A\in g$, $|A|=|E|$. ($|A|$ is the base number of A)

Lemma 1.2[2] For any $A,B\in g$, if $A\cap B\neq\varnothing$, then $|A\cap B|=|E|$.

Lemma1.3 g is an uniform HX-group iff E is a subgroup of G.

Let G be a group and g an uniform HX-group on G. Here we state some results whose proof are very simple .

Proposion1.4 g is an uniform HX-group iff $E^{-1}=E^{(-1)}$.

Proposion1.5 $G^*\leqslant G$.

Proposion1.6 For any $A\in g$ and $a\in A$,$aE=Ea=A$, i.e , E is a normal subgroup of G* .

Proposion1.7 If aE=bE, then $a,b\in A\in g$.

Proposion1.8 For any $A\in g$, $t\in E$ iff $tA=At=A$.

(1)This work was supported by the NSF of China and Henan (69974006, 994052300 ,2000110021).

Proposion1.9 Suppose that $A\in \mathcal{g}$,If aE=A, then $a^{-1}E=A^{-1}$.

Proposion1.10 Let $A\in \mathcal{g}$,for any $a\in A$,$a^{-1}A=E$.

Proposion1.11 Let $A,B\in \mathcal{g}$,If AB=C, then for any $a\in A$,we have aB=C .

Proposion1.12 Let $B\in \mathcal{g}$ and $a\in A\in \mathcal{g}$, xB=aB iff $x\in A$;

Proposion1.13 Suppose that $A,B\in \mathcal{g}$,if $A\cap B\neq \varnothing$, then A=B .

Proposion1.14 $\mathcal{g}=G^*/E$.

2. The condition of a noempty subset of the powerset P(G) is an uniform HX-group

Definition2.1. Let G be a group and X a non-empty set . A action of G on the set X is a map from $G\times X$ to X, with the image of (g,x) being denoted by g(x),which satisfies the following conditions:

(1)e(x)=x, for every $x\in X$,

(2)$g_1g_2(x)=g_1(g_2(x))$, for every g_1, $g_2\in G$ and $x\in X$.

Let G be a group and $\mathcal{g}$ an uniform HX-group on G, then $G^*\leqslant G$. and $e\in E\in \mathcal{g}$, let $G^*\times \mathcal{g}\longrightarrow \mathcal{g}$, $(a,A)\longrightarrow a(A)=aA$, then (1) eA=A, for every $A\in \mathcal{g}$,

(2) (ab)A=a(bA), for every a, $b\in G^*$ and $A\in \mathcal{g}$.

Thus the group G^* acts on the HX-group $\mathcal{g}$.

For every $a\in G^*$, defining η_a: $\mathcal{g}\longrightarrow \mathcal{g}$, $\eta_a(A)=aA$, for all $A\in G$.

Since $a^{-1}(aA)= a (a^{-1}A) =eA=A$, η_a is a bijection from $\mathcal{g}$ to $\mathcal{g}$ .Hence $\eta_a\in S(\mathcal{g})$. (S ($\mathcal{g}$) is the symmetric group on $\mathcal{g}$)

Since (ab)A = a(bA) , then $\eta_{ab}=\eta_a\eta_b$, it implies that η: $G^*\longrightarrow S(\mathcal{g})$,η:a $\longrightarrow \eta_a$ is a homomorphism from G^* to $S(\mathcal{g})$.

Conversely, if giving a homomorphism from G^* to $S(\mathcal{g})$ η: $G^*\longrightarrow S(\mathcal{g})$, we have a action from G^* to $\mathcal{g}$ by the definition: $a(A)=\eta_a(A)=aA$, for all $a\in G^*$ and $A\in \mathcal{g}$. Hence the kernel of the homomorphism η , denoted by ker η , is called the kernel of the action from G^* to $\mathcal{g}$.

When the group G^* acts on the $\mathcal{g}$, we define that $A\sim B$ iff there exists $a\in G^*$ such that B=aA.Then $\sim$ is a equivalent relation on $\mathcal{g}$ and the equivalent class which contains A is denote G^*A. And $G^*A=\{ aA \mid a\in G^*, A\in \mathcal{g} \}$ is called the G^*-orbit of A.It is obviously that

$$\mathcal{g}=\bigcup_{A\in \mathcal{g}} G^*A$$

A is called the fixed element of G^* if $G^*A=\{ A \}$, and we denote the $\{A\in \mathcal{g} \mid aA=A\}$ by F(a).

Definition2.2 The action of G^* on the $\mathcal{g}$ is transitive if $G^*A=\mathcal{g}$.

Let $Stab_{G^*}A=\{a\in G^* \mid aA=A\}$. It is clear that $stab_{G^*}A$ is a subgroup of G^*, is called the stability subgroup of A.

Lemma2.1[8] $| stab_{G^*} A | \cdot | G^*A | = | G^* |$.

Lemma2.2[8] (Burnside's Lemma) Let t be the number of orbits of G^* acts on $\mathcal{g}$, then

(1)This work was supported by the NSF of China and Henan (69974006, 994052300 ,2000110021).

$$t\,|G^*| = \sum_{a\in G^*}|F(a)|\ ,$$

When the action of G^* on $\mathcal{g}$ is transitive, then $t=1$ thus $|G^*| = \sum_{a\in G^*}|F(a)|$

Theorem2.1 Let G^* be a subgroup of G and $\mathcal{g}$ an uniform HX-group of G, then

$$\ker\eta = E.$$

Proof Suppose $\eta: G^* \to S(\mathcal{g})$, $\eta: a \to \eta_a$ is a group homomorphism, and for every $A\in\mathcal{g}$, $\eta_a(A)=aA$, then for all $a\in\ker\eta$, $\eta_a=1$ (1 is the identity transformation), $\eta_a(A)=1(A)=A=aA$ implies $a\in E$.

Conversely, $\eta_a(A)=aA=A=1(A)$ for every $a\in E$ and $A\in\mathcal{g}$, it implies that $\eta_a=1$, thus $a\in \ker\eta$ .Hence $\ker\eta=E$.

Theorem2.2 Let G be a group and $\mathcal{g}$, a noempty subset of the powerset P(G), $G^*=\cup\{A \mid A\in\mathcal{g}\}$. Then $\mathcal{g}$ is an uniform HX-group on G if and only if the following conditions are satisfied:

(1) $G^*\leqslant G$;

(2) there exists E in $\mathcal{g}$, such that E is a normal subgroup of G^*;

(3) the action of G^* on $\mathcal{g}$:$(a,A) \to aA$ (for $a\in G^*$ and $A\in\mathcal{g}$) is transitive.

Proof (Necessary) Ler $\mathcal{g}$ be an uniform HX-group of G. By proposion 1.5, 1.6 , G^* is a subgroup of G and if E is the identity of $\mathcal{g}$, then E is a normal subgroup of G^*. For every $a\in G^*$ and $A\in\mathcal{g}$, defining $(a,A)\to aA$,then G^* acts on $\mathcal{g}$.

Suppose that t is the number of orbits of G^* acts on $\mathcal{g}$, by lemma2.2 , $t|G^*|= \sum_{a\in G^*}|F(a)|$

Since $\mathcal{g}$ is a group,by the definition of F(a) , when $a\notin E$, $F(a)=\varnothing$ and when $a\in E$, $F(a)=\mathcal{g}$, Thus

$$t\cdot|G^*| = |E|\cdot|F(a)| = |E|\cdot|\mathcal{g}| = |G^*|$$

Hence $t=1$, the action of G^* on $\mathcal{g}$ is transitive.

(Sufficiency) Suppose that the conditions (1), (2), (3) are satisfied.For every $A\in\mathcal{g}$, $\mathcal{g}=G^*A=\{aA \mid a\in G^*\}$, then for every $B\in\mathcal{g}$, there exixts $a\in G^*$ such that $aA=B$, it implies $|A|=|B|$.

Since E is a normal subgroup of G^*, and $\mathcal{g}=G^*E=\{aE \mid a\in G^*\}$, if $A\in\mathcal{g}$, then there exists $a\in G^*$, such that $A=aE$ and $a\in A$. We have

$$AE=(aE)E=aE=A,\ EA=E(aE)=(Ea)E=(aE)E=A..$$

Defining $A\cdot B=\{ab|a\in A, b\in B\}$, is denote by AB. Since $\mathcal{g}=G^*E$, then $A=aE$, $B=bE$, $AB=aE\cdot bE= a(Eb)E=(abE)E=(ab)E \in \mathcal{g}$,hence " • " is an algebric operation $\mathcal{g}$, and $AB=\bigcup_{a\in A} aB = \bigcup_{b\in B} Ab$.

Moreover, for every $A\in\mathcal{g}$, $A=aE$, there exists $a^{-1}\in G^*$ such that $a^{-1}E\in\mathcal{g}$, $(a^{-1}E)(aE)=(a^{-1}a)E=eE=E$, $A^{-1}=a^{-1}E$.Hence, $\mathcal{g}$ is a HX-group on G.

Suppose that $x\in A=aE=Ea$,then $A^{-1}=a^{-1}E$, $x=ta$ $(t\in E)$, $x^{-1}=a^{-1}t^{-1}\in a^{-1}E$, hence $A^{(-1)}\subseteq A^{-1}$.

On other hand,for any $x\in A^{-1}$, $x=a^{-1}t=t_1a^{-1}=(at_1^{-1})^{-1}\in A^{(-1)}$ (where $t_1\in E$), $A^{-1}\subseteq A^{(-1)}$, it implies that $A^{-1}=A^{(-1)}$, $\mathcal{g}$ is an uniform HX-group on G.

(1)This work was supported by the NSF of China and Henan (69974006, 994052300 ,2000110021).