Semi-preseparation Axioms in fts *

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Abstract

In this paper, we introduce some new separation axioms–fuzzy semi-preseparation axioms, and also establish some of their characteristic properties.
Keywords: *Fuzzy topology, semi-preopen set, semi-pre-q-neighborhood, semi-preirresolute mapping, semi-pre-T_i axioms.*

1 Introduction and Preliminaries

Recently, Thakur and Singh introduced fuzzy semi-preopen sets in[10]. In this paper, we follow the concept of fuzzy semi-preopen set, with the help of fuzzy semi-pre-p-neighborhoods[10], to introduce fuzzy semi-preirresolute mapping and fuzzy semi-preseparation axioms and establish some of their characteristic properties.

In this paper, by X and Y we mean fuzzy topological spaces(fts, for short). For two fuzzy sets A and B in X, we write AqB to mean that A is quasi-coincident with B, i.e., there is at least one point $x \in X$ such that $A(x)+B(x) > 1$. Negation of such a statement is denoted as $A \not{q} B$. B is said to be a quasi-neighbourhood of A iff there exists a fuzzy open set U such that $AqU \leq B$[8]. The constant fuzzy sets taking on the values 0 and 1 on X are designated by 0_X and 1_X respectively. For a fuzzy set A in X, the notations A^o, A^-, A' and $suppA$ will respectively stand for the interior, closure, complement and support of A. A fuzzy set A in X is said to be (1) fuzzy semi-preopen if there is a fuzzy preopen set B such that $B \leq A \leq B^-$; (2) fuzzy semi-preclosed if there is a fuzzy preclosed set B such that $B^o \leq A \leq B$[10]. The family of fuzzy semi-preopen (resp. semi-preclosed) sets of a fts X will be denoted by $\psi(X)(resp.\psi'(X))$. Let A be a fuzzy set in X. Then $A^\square = \bigcup\{B : B \leq A, B \in \psi(X)\}$ and $A^\frown = \bigcap\{B : A \leq B, B \in \psi'(X)\}$ are called

*Project supported by the National Natural Science Foundation of China

the semi-preinterior and semi-preclosure of A, respectively. A fuzzy set A is called a fuzzy semi-preneighborhood of a fuzzy point x_α in X if there exists a $B \in \psi(X)$ such that $x_\alpha \in B \leq A$. A fuzzy set A is called a fuzzy semi-pre-q-neighborhood of a fuzzy point x_α in X if there exists a $B \in \psi(X)$ such that $x_\alpha q B \leq A$[10]. The set of all semi-preneighborhoods(semi-pre-q-neighborhoods) of x_α is denoted by $\xi(x_\alpha)(\eta(x_\alpha))$.

We easily prove that $A^{\square\prime} = A'^{\frown}$ and $A^{\frown\prime} = A'^{\square}$ for a fuzzy set A in X.

2 Fuzzy Semi-preirresolute Mappings

Definition 2.1. A mapping $f : X \to Y$ is called fuzzy semi-preirresolute if $f^{-1}(B) \in \psi(X)$ for each $B \in \psi(Y)$.

Theorem 2.2. For a mapping $f : X \to Y$ the following are equivalent:

(1) f is fuzzy semi-preirresolute.

(2) $f^{-1}(B) \in \psi'(X)$ for each $B \in \psi'(Y)$.

(3) $f(A^\frown) \leq (f(A))^\frown$ for each fuzzy set A in X.

(4) $(f^{-1}(B))^\frown \leq f^{-1}(B^\frown)$ for each fuzzy set B in Y.

(5) $f^{-1}(B^\square) \leq (f^{-1}(B))^\square$ for each fuzzy set B in Y.

(6) For each fuzzy point x_α in X and each $V \in \psi(Y)$ with $f(x_\alpha) \in V$, there exists a $U \in \psi(X)$ such that $x_\alpha \in U$ and $f(U) \leq V$.

(7) For each fuzzy point x_α in X and each $V \in \psi(Y)$ satisfying $f(x_\alpha) q V$ there exists a $U \in \psi(X)$ such that $x_\alpha q U$ and $f(U) \leq V$.

Theorem 2.3. Let $f : X \to Y$ be one-to-one and onto. f is a fuzzy semi-preirresolute mapping iff $(f(A))^\square \leq f(A^\square)$ for each fuzzy set A in X.

3 Fuzzy Semi-preseparation Axioms

Definition 3.1. An fts X is called fuzzy semi-pre-T_0 iff for every pair of distinct fuzzy points x_α and y_β the following conditions are satisfied:

(1) When $x \neq y$, either there is a $U \in \xi(x_\alpha)$ such that $U \not q y_\beta$ or there is a $V \in \xi(y_\beta)$ such that $V \not q x_\alpha$.

(2) When $x = y$ and $\alpha < \beta$(say), there is a $V \in \eta(y_\beta)$ such that $V \not q x_\alpha$.

Theorem 3.2. An fts X is fuzzy semi-pre-T_0 iff for every pair of distinct fuzzy points x_α and y_β, either $x_\alpha \notin (y_\beta)^\frown$ or $y_\beta \notin (x_\alpha)^\frown$.

Proof. Let X be fuzzy semi-pre-T_0 and x_α and y_β be two distinct fuzzy points in X. When $x \neq y$, there is a $U \in \xi(x_1)$ such that $U \not{q} y_\beta$ or there is a $V \in \xi(y_1)$ such that $V \not{q} x_\alpha$. Suppose there is a $U \in \xi(x_1)$ such that $U \not{q} y_\beta$. Then $U \in \eta(x_\alpha)$ and $U \not{q} y_\beta$. Hence $x_\alpha \notin (y_\beta)^\frown$. When $x = y$ and $\alpha < \beta$(say), then there is a $V \in \eta(y_\beta)$ such that $V \not{q} x_\alpha$ and so in this case also $y_\beta \notin (x_\alpha)^\frown$.

Conversely, let x_α and y_β be two distinct fuzzy points in X. We suppose, without loss of generality, that $x_\alpha \notin (y_\beta)^\frown$. When $x \neq y$, since $x_\alpha \notin (y_\beta)^\frown, x_1 \notin (y_\beta)^\frown$ and hence $(y_\beta)^{\frown\prime}(x) = 1$. Then $(y_\beta)^{\frown\prime} \in \xi(x_\alpha)$ such that $(y_\beta)^{\frown\prime} \not{q} y_\beta$. When $x = y$ we much have $\alpha > \beta$ and then there is a $U \in \eta(x_\alpha)$ such that $U \not{q} y_\beta$.

Definition 3.3. An fts X is called fuzzy semi-pre-T_1 iff for every pair of distinct fuzzy points x_α and y_β the following conditions are satisfied:

(1) When $x \neq y$, there are $U \in \xi(x_\alpha)$ and $V \in \xi(y_\beta)$ such that $U \not{q} y_\beta$ and $V \not{q} x_\alpha$.

(2) When $x = y$ and $\alpha < \beta$(say), there is a $V \in \eta(y_\beta)$ such that $V \not{q} x_\alpha$.

Obviously, fuzzy semi-pre-T_1 $\Rightarrow$fuzzy semi-pre-T_0.

Theorem 3.4. An fts X is fuzzy semi-pre-T_1 iff every fuzzy point x_α is fuzzy semi-preclosed in X.

Proof. Let X be fuzzy semi-pre-T_1 and x_α and y_β be two distinct fuzzy points in X. When $x \neq y$, there are $U, V \in \psi(X)$ and $U \in \xi(x_\alpha)$, $V \in \xi(y_\beta)$ such that $U \not{q} y_\beta$ and $V \not{q} x_\alpha$. Then $x_\alpha \in V'$. Since $V' \in \psi'(X), (x_\alpha)^\frown \leq V'$, equivalent $V \not{q} (x_\alpha)^\frown$. Thus $(x_\alpha)^\frown \leq x_\alpha$, i.e., $x = (x_\alpha)^\frown$. Hence every fuzzy point x_α is fuzzy semi-preclosed in X. When $x = y$, it is analogous to the proof of it above.

Conversely, let x_α and y_β be two distinct fuzzy points in X. When $x \neq y$, since x_α and y_β are fuzzy semi-preclosed in X, $(x_\alpha)'$ and $(y_\beta)'$ are fuzzy semi-preopen. Then $(x_\alpha)' \in \xi(y_\beta)$ and $(y_\beta)' \in \xi(x_\alpha)$ such that $(x_\alpha)' \not{q} x_\alpha$ and $(y_\beta)' \not{q} y_\beta$. When $x = y$ and $\alpha < \beta$(say), obviously $(x_\alpha)' \in \eta(y_\beta)$ such that $(x_\alpha)' \not{q} x_\alpha$.

Definition 3.5. An fts X is called fuzzy semi-pre-T_2 iff for every pair of distinct fuzzy points x_α and y_β the following conditions are satisfied:

(1) When $x \neq y$, there are $U \in \xi(x_\alpha)$ and $V \in \xi(y_\beta)$ such that $U \not{q} V$.

(2) When $x = y$ and $\alpha < \beta$(say), there are $U \in \xi(x_\alpha)$ and $V \in \eta(y_\beta)$ such that $U \not{q} V$.

Obviously,fuzzy semi-pre-T_2 $\Rightarrow$fuzzy semi-pre-T_1. Also, fuzzy semi-T_i[6] $\Rightarrow$ fuzzy semi-pre-T_i, $i = 0, 1, 2$.

Theorem 3.6. An fts X is fuzzy semi-pre-T_2 iff for every fuzzy point x_α in X,

$x_\alpha = \bigcap\{V^\frown : V \in \xi(x_\alpha)\}$

and for any $x, y \in X$ with $x \neq y$, there is a $U \in \xi(x_1)$ such that $y \notin supp(U^\frown)$.

Proof. Let X be fuzzy semi-pre-T_2, x_α and y_β be fuzzy points in X such that $y_\beta \notin (x_\alpha)$. To establich the required equality, it suffices to show the existence of a $V \in \xi(x_\alpha)$ such that $y_\beta \notin V^\frown$. If $x \neq y$, then there are $U, V \in \psi(X)$ and $y_1 \in U, x_\alpha \in V$ such that $U \not{q} V$. Then $V \in \xi(x_\alpha)$ and $U \in \eta(y_\beta)$ such that $U \not{q} V$. Hence $y_\beta \notin V^\frown$. If $x = y$, then $\beta > \alpha$, and hence there are $U \in \eta(y_\beta)$ and $V \in \xi(x_\alpha)$ such that $U \not{q} V$. Then $y_\beta \notin V^\frown$. Finally, for two distinct points x, y of X, since X is fuzzy semi-pre-T_2, there exist $U, V \in \psi(X)$ such that $x_1 \in U, y_1 \in V$ and $U \not{q} V$. Then $V'(y) = 0$ and $U \leq V'$. Since $V' \in \psi'(X), U^\frown \leq V'$. Thus $(U^\frown)(y) = 0$, i.e., $y \notin supp(U^\frown)$.

Conversely, let x_α and y_β be two distinct fuzzy points in X. When $x \neq y$, we first suppose that at least one of α and β is less than 1, say $0 < \alpha < 1$. There exists a positive real number λ with $0 < \alpha + \lambda < 1$. By hypothesis, there exists a $U \in \xi(y_\beta)$ such that $x_\lambda \notin U^\frown$. Then there is a $V \in \psi(X)$ and $V \in \eta(x_\lambda)$ such that $V \not{q} U$. Now, $\lambda + V(x) > 1$ so that $V(x) > 1 - \lambda > \alpha$ and hence $V \in \xi(x_\alpha)$ such that $U \not{q} V$, where $U \in \xi(y_\beta)$. In case $\alpha = \beta = 1$, by hypothesis there is a $U \in \xi(x_1)$ such that $(U^\frown)(y) = 0$. Then $V = U^{\frown\prime} \in \xi(y_1)$ such that $U \not{q} V$. When $x = y$ and $\alpha < \beta$(say), then there is a $U \in \xi(x_\alpha)$ such that $y_\beta \notin U^\frown$. Consequently, there exists a $V \in \eta(y_\beta)$ such that $U \not{q} V$. Considering the above cases, we conclude that X is fuzzy semi-pre-T_2.

Lemma 3.7. Let X be an fts and $U, V \in \psi(X)$. If $U \not{q} V$, then $U^\frown \not{q} V$.

Definition 3.8. An fts X is called fuzzy sp-regular iff for each fuzzy point x_α in X and each $U \in \psi(X)$ with $U \in \eta(x_\alpha)$, there is a $V \in \psi(X)$ and $V \in \eta(x_\alpha)$ such that $V^\frown \leq U$.

Theorem 3.9. For an fts X, the following are equivalent:

(1) X is fuzzy sp-regular.

(2) For each fuzzy point x_α in X and each $B \in \psi'(X)$ with $x_\alpha \notin B$, there is a $U \in \psi(X)$ such that $x_\alpha \notin U^\frown$ and $B \leq U$.

(3) For each fuzzy point x_α in X and each $B \in \psi'(X)$ with $x_\alpha \notin B$, there exist $U, V \in \psi(X)$ such that $U \in \eta(x_\alpha), B \leq V$ and $U \not{q} V$.

(4) For any fuzzy set A and any $B \in \psi'(X)$ with $A \not\leq B$, there are $U, V \in \psi(X)$ such that $AqU, B \leq V$ and $U \not{q} V$.

(5) For any fuzzy set A and any $U \in \psi(X)$ with AqU, there is a $V \in \psi(X)$ such that $AqV \leq V^{\frown} \leq U$.

Proof. (1) $\Rightarrow$ (2) : Let x_α be a fuzzy point in X and $B \in \psi'(X)$ with $x_\alpha \notin B$. Then $B' \in \eta(x_\alpha)$ and $B' \in \psi(X)$. Since X is fuzzy sp-regular, there is a $V \in \psi(X)$ and $V \in \eta(x_\alpha)$ such that $V^{\frown} \leq B'$. Put $U = V^{\frown\prime}$. Then $U \in \psi(X)$ and $V^{\frown\square} \in \eta(x_\alpha)$. Hence,

$$x_\alpha \notin V^{\frown\square\prime} = V^{\frown\prime\frown} = U^{\frown},$$

and $B \leq V^{\frown\prime} = U$.

(2) $\Rightarrow$ (3) : For any fuzzy point x_α in X and any $B \in \psi'(X)$ with $x_\alpha \notin B$, by (2) there is a $U \in \psi(X)$ such that $x_\alpha \notin U^{\frown}$ and $B \leq U$. Hence $U^{\frown\prime} \in \eta(x_\alpha)$ and $U^{\frown\prime} \not{q} U$, where $U^{\frown\prime} \in \psi(X)$. Thus (3) is obtained.

(3) $\Rightarrow$ (4) : Let A be a fuzzy set and $B \in \psi'(X)$ with $A \not\leq B$. Then there is at least one fuzzy point $x_\alpha \in A$ such that $x_\alpha \notin B$. By (3) there are $U, V \in \psi(X)$ such that $U \in \eta(x_\alpha), B \leq V$ and $U \not{q} V$. Since $x_\alpha \in A$, we have AqU.

(4) $\Rightarrow$ (5) : For any fuzzy set A and any $U \in \psi(X)$, AqU implies that $A \not\leq U'$, where $U' \in \psi'(X)$. By (4), there are $V, W \in \psi(X)$ such that $AqV, U' \leq W$ and $V \not{q} W$. Then $V^{\frown} \not{q} W$ by Lemma 3.7. Hence, $AqV \leq V^{\frown} \leq W' \leq U$.

(5) $\Rightarrow$ (1) : Obvious.

Theorem 3.10. Let $f : X \to Y$ be a one-to-one fuzzy semi-preirresolute mapping. If Y is fuzzy semi-pre-T_i then so is X, for $i = 0, 1, 2$.

Proof. We prove only $i = 1$.

Let x_α and y_β be two distinct fuzzy points in X. When $x \neq y$, we have $f(x) \neq f(y)$. Since Y is fuzzy semi-pre-T_1, there are $U, V \in \psi(Y)$ and

$$U \in \xi((f(x))_\alpha), V \in \xi((f(y))_\beta)$$

such that

$$V \not{q}(f(x))_\alpha \text{ and } U \not{q}(f(y))_\beta.$$

Then $f^{-1}(U) \in \xi(x_\alpha)$ and $f^{-1}(V) \in \xi(y_\beta)$ such that

$$f^{-1}(V) \not{q} x_\alpha \text{ and } f^{-1}(U) \not{q} y_\beta.$$

When $x = y$ and $\alpha < \beta$(say), the $f(x) = f(y)$. Since Y is semi-pre-T_1, there is a $V \in \psi(Y)$ and $V \in \eta((f(y))_\beta)$ such that $V \not{q}(f(x))_\alpha$. Then $f^{-1}(V) \in \eta(y_\beta)$ such that $f^{-1}(V) \not{q} x_\alpha$. Thus X is fuzzy semi-pre-T_1.

References

[1] Bai Shi-Zhong, Fuzzy weak semicontinuity, *Fuzzy Sets and Systems* 47(1992) 93-98.

[2] Bai Shi-Zhong, Fuzzy strongly semiopen sets and fuzzy strong semicontinuity, *Fuzzy Sets and Systems* 51(1992) 345-351.

[3] Bai Shi-Zhong, Fuzzy S-irresolute mappings, *J.Fuzzy Math.*4(1996) 397-411.

[4] Bai Shi-Zhong, Q-convergence of nets and weak separation axioms in fuzzy lattices, *Fuzzy Sets and Systems* 88(1997) 379-386.

[5] S.Ganguly and S.Saha, On separation axioms and T_i-fuzzy continuity, *Fuzzy Sets and Systems* 16(1985) 265-275.

[6] B.Ghosh, Semi-continuous and semi-closed mapping and semi-connectedness in fuzzy setting, *Fuzzy Sets and Systems* 35(1990) 345-355.

[7] B.Hutton and I.Reilly, Separation axioms in fuzzy topological spaces, *Fuzzy Sets and Systems* 3(1980) 93-104.

[8] P.M.Pu and Y.M.Liu, Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence, *J.Math.Anal.Appl.*76(1980) 571-599.

[9] M.K.Singal and N.Prakash, Fuzzy preopen sets and fuzzy preseparation axioms, *Fuzzy Sets and Systems* 44(1991) 273-281.

[10] S.S.Thakur and S.Singh, On fuzzy semi-preopen sets and fuzzy semi-precontinuity, *Fuzzy Sets and Systems* 98(1998)383-391.