A Kind of Classification of Fuzzy Finite Automata

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Abstract

In this paper, fuzzy finite automata is classified to two kind of basic models: one is the fuzzy finite automaton with initial states and no outputs, the other is the fuzzy finite automaton with outputs and no initial states. Based on this classification, some important types of fuzzy finite automaton are classified. In the meantime, The relations among these fuzzy finite automata in same models are obtained.

Keywords: Fuzzy finite automata; Fuzzy finite-state automata; Classification.

1. Introduction

The theory of fuzzy sets was introduced by L.A.Zadeh in 1965[1]. The mathematical formulation of a fuzzy automaton was first proposed by W.G.Wee in 1967[2]. E.S.Santos defined so-called maximin automata in 1968[3]. E.T.Lee and L.A.Zadeh gave the concept of fuzzy finite-state automaton in 1969[4]. A fuzzy automaton with a fuzzy initial state was first considered by M.Mizumoto, J.Toyota and K.Tanaka in 1969[5]. Two classes of fuzzy automata corresponding to the Mealy and Moore type of ordinary automata are formulated by K.Asai and S.Kitajima in 1971[6]. D.S.Malik, J.N.Mordeson and M.K.Sen also introduce a kind of fuzzy finite automaton in 1999[7]. In the following, we shall show the relations among these fuzzy finite automata and obtain our conclusions.

Notes: Throughout this paper, we only consider stationary fuzzy transition function. That is to say, these fuzzy transition functions is independent of time.

2. Preliminaries

Let's give some kinds of important definitions of fuzzy finite automaton.

(1) *W.G.Wee* [8]

A (finite) fuzzy automaton is a quintuple A:

$$A=(I, V, Q, f, g)$$

I: nonempty finite set of objects (input states)

V: nonempty finite set of objects (output states)

Q: nonempty finite set of objects (internal states)

f: membership function of a fuzzy set in $Q\times I\times Q$; i.e., f: $Q\times I\times Q\to[0,1]$

g: membership function of a fuzzy set in $V\times I\times Q$; i.e., g: $V\times I\times Q\to[0,1]$.

Let I_j be an input sequence of length j, then

$$f_A(q_l, I_j, q_m) = \max_{q_o, q_p, \cdots, q_s\in Q}\{\min[f_A(q_l, i_1, q_o), f_A(q_o, i_2, q_p), \ldots, f_A(q_s, i_j, q_m)]\}$$

(2) *E. S. Santos* [3]

A maximin automaton A is a system:

$$A=(U, S, \mu^*, F, \mu_h)$$

U: a finite nonempty set of inputs;

S: a finite nonempty set of states;

μ^*: $S\times U\times S\to[0,1]$, called the state transition function;

F: the set of final states;

μ_h: $S\to[0,1]$, denoting the initial distribution;

$$\mu^*(s, \Lambda, s') = 1, \quad \text{if} \quad s=s'$$
$$= 0, \quad \text{if} \quad s\neq s'$$

$$\mu^*(s, u^*u, s') = \max_{s''\in S}\min[\mu^*(s, u^*, s''), \mu^*(s'', u, s')]$$

(3) *E.T.Lee* and *Zadeh* [4]

(a) A determinable fuzzy finite-state automaton *FA* is:

$$FA = (Q, \Sigma, \delta, q_0, F)$$

Q: a finite set of states;

Σ: a finite set of input symbol;

δ: a mapping from $Q\times\Sigma$ to $Q\times[0,1]$. δ is called state transition function;

δ: $Q\times\Sigma\to Q\times[0,1]$

$(q,a)\to\delta(q,a)=(q',\mu)$

$q_0\in Q$: the initial state;

$F\subseteq Q$: a set of final states.

(b) A nondeterminable fuzzy-state automaton *FA* is:

$$FA = (Q, \Sigma, \delta, q_0, F)$$

δ: $Q\times\Sigma\to 2^{Q\times[0,1]}$

A determinable and nondeterminable fuzzy finite-state automaton are all called fuzzy finite-state automata.

δ: $Q\times\Sigma^*\to Q\times[0,1]$

$\delta(q,e)=(q,1)$, $\delta(q,xa)=\delta(q_1,x)$, $\delta(q,a)=(q_1,\mu)$, $a\in\Sigma$, $x\in\Sigma^*$.

(4) *M. Mizumoto, J. Toyoda* and *K. Tanaka* [5]

A finite fuzzy automaton over the alphabet Σ is a system A:

$$A=(Q, \pi, \{F(\sigma)|\sigma\in\Sigma\}, \eta^G)$$

$Q=\{q_1,q_2,\cdots,q_n\}$: a non-empty finite set of internal states;

$\pi=(\pi_{q_1},\pi_{q_2},\cdots,\pi_{q_n})$: an n-dimensional fuzzy row vector, $0\le\pi_{q_i}\le1$, $i=1(1)n$; π is called the initial state designator;

$G\subseteq Q$: the set of final states;

$\eta^G=(\eta_{q_1},\eta_{q_2},\cdots,\eta_{q_n})'$: an n-dimensional column vector; if $q_i\in G$, then $\eta_{q_i}=1$; otherwise $\eta_{q_i}=0$; η^G is called the set of final states;

$\underset{n\times n}{F(\sigma)}=[f_{q_i,q_j}(\sigma)]_{j=1(1)n}^{i=1(1)n}$: a fuzzy matrix of order n; $F(\sigma)$ is called the fuzzy transition matrix of A;

Let element $f_{q_i,q_j}(\sigma)$ of $F(\sigma)$ be $\mu_A(q_i, \sigma, q_j)$ where q_i, $q_j\in Q$, $\sigma\in\Sigma$ and μ_A is a mapping:

μ_A: $Q\times\Sigma\times Q\to[0,1]$ may be called the fuzzy transition function. For an input sequence $x=\sigma_1\sigma_2\ldots\sigma_n\in\Sigma^*$ and $s, t\in Q$,

$$\mu_A(s,x,t)=\max_{\substack{q_i\in Q\\ 1\le i\le m-1}} \min[\mu_A(s,\sigma_1,q_1),\mu_A(q_1,\sigma_2,q_2),\cdots,\mu_A(q_{m-1},\sigma_m,t)]$$

For Λ, $x, y\in\Sigma^*$ and $s, t\in Q$,

$\mu_A(s,\Lambda,t)=1$, if $s=t$

$=0$, if $s\neq t$

$$\mu_A(s,xy,t)=\max_{q\in Q}\min[\mu_A(s,x,q),\mu_A(q,y,t)]$$

(5) *K. Asai* and *S. Kitajima* [6]

(a) Mealy type of fuzzy automata: The Mealy type of fuzzy automata may be expressed as shown in following:

$$M=\{S, X, U, F(u/x)\}$$

$S=\{s_1,s_2,\cdots,s_\nu\}$: set of ν states,

$X=\{x_1,x_2,\cdots,x_\mu\}$: set of μ inputs,

$U=\{u_1,u_2,\cdots,u_\xi\}$: set of ξ outputs,

$F(u/x)=\nu\times\nu$ fuzzy transition matrix.

In the case of single input, the fuzzy transition matrices may be expressed as shown in following: $F(u_h/x_l)=[f_{ij}(u_h/x_l)]$, where $h=1,2,\ldots,\xi$.

$f_{ij}(u_h/x_l)$: membership function that the automaton will go to state s_j from state s_i *and* send out the output u_h when the input x_l is applied, where $i, j = 1, 2, \ldots, \nu$.

The membership function for the transition by a branch leading from a state s_i to a state s_j is shown as:

$$f_{ij}^{\,l}(u/x_l)=\max\{f_{ij}(u_1/x_l), f_{ij}(u_2/x_l), \ldots, \{f_{ij}(u_\xi/x_l)\}$$

By using the symbols of algebraic sum instead of the symbol max, we can write it as:

$$f_{ij}^1(u/x_l) = f_{ij}(u_1/x_l) + f_{ij}(u_2/x_l) + \ldots + f_{ij}(u_\xi/x_l)$$

So the first order fuzzy transition matrix will be:

$$F^1(u/x_l) = F^1(u_1/x_l) + F^1(u_2/x_l) + \ldots + F^1(u_\xi/x_l)$$

The k^{th} order fuzzy transition matrix will be obtained as:

$$F^k(u/x_l) = F^1(u/x_l) \circ F^1(u/x_l) \circ \ldots \circ F^1(u/x_l)$$

In case of multi-input sequence, the membership function for a path in which the k^{th} order transition may be executed from state s_i to state s_j via serial branches is given as:

$$f_{ij}^k = f_{ij}^k\{s_i, x^k, s_j\}$$

$$= \sup_{s_p, s_q, \cdots, s_v} \{min[f_{ip}\{s_i, x(1), s_p\}, f_{pq}\{s_p, x(2), s_q\}, \cdots, f_{vj}\{s_v, x(k), s_j\}]\}$$

$$= \sup_{s_p, s_q, \cdots, s_v} \{min[f_{ip}^1(u/x_1), f_{pq}^1(u/x_2), \cdots, f_{vj}^1(u/x_k)]\}$$

(b) Moore type of fuzzy automata: The Moore type of fuzzy automaton may be expressed as shown in following:

$$M^* = \{S, X, U, F(x), G(s, u)\}$$

$S = \{s_1, s_2, \cdots, s_\nu\}$: set of ν states,

$X = \{x_1, x_2, \cdots, x_\mu\}$: set of μ inputs,

$U = \{u_1, u_2, \cdots, u_\xi\}$: set of ξ outputs,

$F(x)$: $\nu \times \nu$ fuzzy transition matrix,

$G(s, u)$: $\nu \times \xi$ output matrix.

The transition matrix may be shown in following: $F(x_l) = [f_{ij}(x_l)]$, where $l = 1, 2, \ldots, \mu$.

$f_{ij}(x_l)$: membership function for the transition from state s_i to state s_j when an input x_l is applied, where $i, j = 1, 2, \cdots, \nu$.

The output matrix may be shown in following: $G(s, u) = [g_{jh}(u_h)]$, where

$g_{jh}(u_h)$: membership function for the choice of an output u_h at the state s_j when the transition has been executed to the state s_j.

In the Moore type, two methods of transition may be defined as follow:

(i) The k^{th} order transition matrix may be given by

$$F^k(x_l) = F^1(x_l) \circ F^1(x_l) \circ \ldots \circ F^1(x_l)$$

In case of multi-input sequence, the membership function for a path in which the k^{th} order transition may be executed from state s_i to state s_j via serial branches is given as:

$$f_{ij}^k = \sup_{s_p, s_q, \cdots, s_v} \{min[f_{ip}^1(x_1), f_{pq}^1(x_2), \cdots, f_{vj}^1(x_k)]\}$$

Then an output may be selected by the output matrix.

(ii) The membership function for the transition from state s_i to state s_j when a single input x_l is applied may be given as:

$$f_{ij}^1 = \max_h \{min(f_{ij}(x_l), g_{jh}(u_h))\}$$

$$F^1(u/x_l) = [f_{ij}^1],\ F^k(u/x_l) = F^1(u/x_l) \circ F^1(u/x_l) \circ \ldots \circ F^1(u/x_l)$$

(6) *D.S. Malik, J.N. Mordeson* and *M.K. Sen* [7]

A fuzzy finite automaton (ffa) is a five-tuple M:

$$M = (Q, X, Y, \mu, \omega)$$

Q: the states set;

X: the input symbols set;

Y: the output symbols set;

μ: $Q \times X \times Q \to [0,1]$, is called the fuzzy transition function;

ω: $Q \times X \times Y \to [0,1]$, is called the fuzzy output function;

The following conditions hold:

(1) $\forall p, q \in Q, \exists a \in X$, such that $\mu(p, a, q) > 0 \Rightarrow \exists b \in Y$, such that $\omega(p, a, b) > 0$;

(2) $\forall p \in Q, a \in X, \exists b \in Y$, such that $\omega(p, a, b) > 0 \Rightarrow \exists q \in Q$, such that $\mu(p, a, q) > 0$;

(3) $\forall p \in Q,\ a \in X,\ \vee\{\mu(p, a, q) \mid q \in Q\} \geq \vee\{\omega(p, a, b) \mid b \in Y\}$.

μ^*: $Q \times X^* \times Q \to [0,1]$

$\mu^*(p, \lambda, q) = 1$, if $p = q$

$= 0$, if $p \neq q$

$\mu^*(p, xa, q) = \vee\{\mu(p, a, r) \wedge \mu^*(r, x, q) \mid r \in Q\}$ for all $p, q \in Q$, $x \in X^*$ and $a \in X$.

ω^*: $Q \times X^* \times Y^* \to [0,1]$

$\omega^*(p, x, y)=1$, if $x = y = \lambda$
$=0$, if $x\neq\lambda$, $y=\lambda$ or $x=\lambda$, $y\neq\lambda$;
$\omega^*(p, xa, yb)= \vee\{\omega(p, a, b)\wedge\mu(p, a, r)\wedge\omega^*(r, x, y)|r\in Q\}$

3. A kind of classification of fuzzy automata

In fact, we can classify these fuzzy finite automata to two kinds of basic models: one is the fuzzy finite automaton with initial states and no outputs, the other is the fuzzy finite automaton with outputs and no initial states. Based on this classification, (2)-type, (3)-type, (4)-type belong to the former and (1)-type, (5)-(a)-type, (5)-(b)-type, (6)-type are included in the latter. At the same time, we can obtain the relations among these fuzzy finite automata in same models. We have the following conclusions:

Theorem 3.1 For a given (2)-type fuzzy finite automaton $A=(U, S, \mu^*, F, \mu_h)$, there exists a (4)-type fuzzy finite automaton $A^*=(Q, \pi, \{F(\sigma)|\sigma\in\Sigma\}, \eta^F)$ such that $L(A)=L(A^*)$,vice versa.

Proof. Let $A=(U, S, \mu^*, F, \mu_h)$ be a (2)-type fuzzy finite automaton with n states. We can define a (4)-type fuzzy finite automaton $A^*=(Q, \pi, \{F(\sigma)|\sigma\in\Sigma\}, \eta^F)$ in which

$$Q = S, \pi = (\pi_{s_1}, \pi_{s_2}, \cdots, \pi_{s_n}) = (\mu_h(s_1), \mu_h(s_2), \cdots, \mu_h(s_n)), \Sigma = U, G = F$$

$$\forall\sigma\in U^*, \underset{n\times n}{F(\sigma)} = [f_{s_i,s_j}(\sigma)]_{j=1(1)n}^{i=1(1)n} = [\mu_{A^*}(s_i,\sigma,s_j)]_{j=1(1)n}^{i=1(1)n} = [\mu^*(s_i,\sigma,s_j)]_{j=1(1)n}^{i=1(1)n}.$$

Conversely, suppose $A^*=(Q, \pi, \{F(\sigma)|\sigma\in\Sigma\}, \eta^F)$ is a (4)-type fuzzy finite automaton with n states. We can obtain a (2)-type fuzzy finite automaton $A=(U, S, \mu^*, F, \mu_h)$ in which

$$S = Q, \mu_h(q_i) = \pi_{q_i}, U = \Sigma, F = G$$

$$\forall\sigma\in\Sigma^*, \mu^*(q_i,\sigma,q_j) = f_{q_i,q_j}(\sigma) = \mu_{A^*}(q_i,\sigma,q_j)$$

Theorem 3.2 For a given a (3)-type fuzzy finite automaton $FA=(Q, \Sigma, \delta, q_0, F)$,there exists a (4)-type fuzzy finite automaton $A=(Q, \pi, \{F(\sigma)|\sigma\in\Sigma\}, \eta^F)$ such that $L(FA)=L(A)$.

Proof. Let $FA=(Q_{FA}, \Sigma_{FA}, \delta, q_0, F)$ be a (3)-type fuzzy finite automaton, we can obtain a (4)-type fuzzy finite automaton $A=(Q, \pi, \{F(\sigma)|\sigma\in\Sigma\}, \eta^F)$ in which

$$\pi = (\pi_{q_1}, \pi_{q_2}, \cdots, \pi_{q_0}, \cdots, \pi_{q_n}) = (0,0,\cdots,1,\cdots,0), Q = Q_{FA}, \Sigma = \Sigma_{FA}, G = F$$

$$\forall\sigma\in\Sigma^*, \underset{n\times n}{F(\sigma)} = [f_{q_i,q_j}(\sigma)]_{j=1(1)n}^{i=1(1)n} = [\mu_{A^*}(q_i,\sigma,q_j)]_{j=1(1)n}^{i=1(1)n} = [\delta(q_i,\sigma,q_j)]_{j=1(1)n}^{i=1(1)n}$$

Theorem 3.3 [6] Every (5)-(a)-type fuzzy finite automaton $M=\{S, X, U, F(u/x)\}$ can transfer to a (5)-(b)-(ii)-type fuzzy finite automaton $M^*=\{S, X, U, F(x), G(s, u)\}$, vice versa.

Theorem 3.4 Every (1)-type fuzzy finite automaton $A=(I, V, Q, f, g)$ with n internal states and ξ output states can transfer to a (5)-(b)-(i)-type fuzzy finite automaton $M^*=\{S, X, U, F(x), G(s, u)\}$.

Proof. Let $A=(I, V, Q, f, g)$ be a (1)-type fuzzy finite automaton, we can construct a (5)-(b)-(i)-type fuzzy finite automaton $M^*=\{S, X, U, F(x), G(s, u)\}$ in which:

$$S = Q, X = I, U = V, F(x) = [f_{ij}(x_l)]_{n\times n} = [f(q_i, I_l, q_j)]_{n\times n}, where I_l \in I$$

$$G(s,u) = [g_{jh}(u_h)]_{n\times\xi} = [\vee_{I_l\in I} g(v_h, I_i, q_j)]_{n\times\xi}$$

Theorem 3.5 Every a (6)-type fuzzy finite automaton $M_6=(Q, X_6, Y, \mu, \omega)$ with n states and ξ output symbols can transfer to a (5)-(a)-type fuzzy finite automaton $M=\{S, X, U, F(u/x)\}$.

Proof. Let $M_6=(Q, X_6, Y, \mu, \omega)$ be a (6)-type fuzzy finite automaton, we can get a (5)-(a)-type fuzzy finite automaton $M=\{S, X, U, F(u/x)\}$ in which:

$$S = Q, X = X_6, U = Y$$

$$F^1(y/x_l) = [f_{ij}^1(y/x_l)]_{n\times n}$$

$$= [\vee_{k=1}^{\xi} f_{ij}(y_k/x_l)]_{n\times n}$$

$$= [\vee_{k=1}^{\xi}(\mu(q_i, x_l, q_j) \wedge \omega(q_i, x_l, y_k))]_{n\times n}$$

4. Conclusion

In the preceding paragraphs, fuzzy finite automata was classified two kinds of basic models: one has initial states and no outputs, the other has outputs and no initial states. In the former the interest is directed to the languages accepted by various automata, but in the latter the interest is directed to automata "behaviour" and "simulations". The "behaviour" of an automaton is described by its "responses" to various experiments (expressed as sequences of input symbols). An automaton *A* simulates the behaviour of an automaton *B* in case *A* can perform any computation *B* can perform and the outputs produced will be the same. Further, some important types of fuzzy finite automaton in same class are compared and have some relations that are obtained in this paper.

In spite of which models a fuzzy finite automaton, there are same problems such as the minimization of fuzzy finite automata, minimization algorithm of fuzzy finite automata etc. Because of the above conclusions, We shall research the minimization of fuzzy finite automata and its algorithm on (4)-type and (5)-(b)-type fuzzy finite automata in further.

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