Homomorphism on the fuzzy space rings

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Abstract: We defined the concept of Homomorphism and Isormorphism to the fuzzy space rings and studied their properties.

1. Preliminary

In this section, unless otherwise stated, I always represents the closed unit interval [0,1] of real numbers; L, K denote arbitary lattices. L K indicates the lattice L×K with the partial order defined by

(i) $(r_1,r_2)\le(s_1,s_2)$, *iff* $r_1\le s_1$ *and* $r_2\le s_2$, *where* $s_1\ne 0$ *and* $s_2\ne 0$,

(ii) $(0,0)=(s_1,s_2)$, *whenever* $s_1=0$ *or* $s_2=0$.

Definition1.1 Let X be an ordinary set and L be a completely distributive lattice with maximal and minimal elements denoted by 1,0, respectively. The fuzzy space, denoted by (X,L), is defined as follows:

$$(X,L)=\{(X,L);x\in X\},$$

Where (X,L) is called a fuzzy element and it is given by the relation

$$(x,L)=\{(x,r);r\in L\}.$$

The sublattice $l\subset$L is called an M-sublattice of L, if it has at least one element more than 0 and a maximal element denoted by 1_l.

Definition1.2 A fuzzy binary operation $\underset{\sim}{F}=(F,f_{xy})$ on the fuzzy space (X,I) is a fuzzy function from (X,I) (X,I) to (X,I), i.e.

$\underset{\sim}{F}=(F,f_{xy})$: (X×X,II)→(X,I), where F:X×X→X with onto comembership functions f_{xy}: II→I which satisfy $f_{xy}(r,s)\ne 0$ if r≠0 and s≠0.

In the following we shall use the notions:

$$F(x,y)=xFy \text{ and } f_{xy}(r,s)=rf_{xy}s.$$

Thus, for any two fuzzy elements (x,I), (y,I) of (X,I), we have

(x,I)$\underset{\sim}{F}$(y,I)=:$\underset{\sim}{F}$((x,I)(y,I))= $\underset{\sim}{F}$((x,y),II)=(F(x,y), f_{xy}(II))=:(xFy,I).

It is clear that F is a binary operation on X.

The operation binary operation $\underset{\sim}{F}=(F,f_{xy})$ on (X,I) is said to be uniform if the associated comembership functions f_{xy} are identical for all x,y∈X.

Definition1.3 Let $\underset{\sim}{F}^+=(F^+,f_{xy}^+)$ and $\underset{\sim}{F}^\bullet=(F^\bullet,f_{xy}^\bullet)$ are two fuzzy binary operations on the fuzzy space (X,I). We call $((X,I),\underset{\sim}{F}^+,\underset{\sim}{F}^\bullet)$ a fuzzy ring if following conditions holds:

(i) $((X,I),\underset{\sim}{F}^+)$ is a fuzzy Abelian group,

(ii) $((X,I),\underset{\sim}{F}^\bullet)$ is a fuzzy semigroup,

(iii) The distributive laws

$$(x,I)\underline{E}^{\bullet}((y,I)\underline{E}^{+}(z,I)) = ((x,I)\underline{E}^{\bullet}(y,I))\underline{E}^{+}((x,I)\underline{E}^{\bullet}(z,I)),$$
$$((y,I)\underline{E}^{+}(z,I))\underline{E}^{\bullet}(x,I) = ((y,I)\underline{E}^{\bullet}(x,I))\underline{E}^{+}((z,I)\underline{E}^{\bullet}(x,I))$$

holds for all (x,I), (y,I), (z,I)∈(X,I). The fuzzy ring $((X,I),\underline{E}^{+},\underline{E}^{\bullet})$ is uniform if $\underline{E}^{+}$ and $\underline{E}^{\bullet}$ are all-uniform.

Proposition1.1 Let $((X,I),\underline{E}^{+},\underline{E}^{\bullet})$ be a fuzzy ring, then $(X,F^{+},F^{\bullet})$ is an ordinary ring, and $((X,I),\underline{E}^{+},\underline{E}^{\bullet})$ is isomorphic to $(X,F^{+},F^{\bullet})$ under the correspondence $\varphi : x \to (x.I)$.

Definition1.4 The fuzzy subspace $U = \{(x,u_x); x \in U_0\}$ is said to be a fuzzy subring of the fuzzy ring $((X,I),\underline{E}^{+},\underline{E}^{\bullet})$, if

(i) $\underline{E}^{+}$ and $\underline{E}^{\bullet}$ are closed on the fuzzy subspace U,

(ii) $(U,\underline{E}^{+},\underline{E}^{\bullet})$ satisfies the axioms of the ordinary ring.

Proposion1.2 The fuzzy subspace $U = \{(x,u_x); x \in U_0\}$ is a fuzzy subring of the fuzzy ring $((X,I),\underline{E}^{+},\underline{E}^{\bullet})$ iff

(i) $(U_0,F^{+},F^{\bullet})$ is an ordinary subring of $(X,F^{+},F^{\bullet})$,

(ii) $f^{+}_{xy}(u_x,u_y) = u_{xF^{+}y}$, $f^{\bullet}_{xy}(u_x,u_y) = u_{xF^{\bullet}y}$, $x,y \in U_0$.

Definition1.5 If $U = H_0(A)$, $U = \underline{H}(A)$ and $U = \overline{H}(A)$ are fuzzy subring of $((X,I),\underline{E}^{+},\underline{E}^{\bullet})$, then we say the fuzzy subset A of X induces fuzzy subring of $((X,I),\underline{E}^{+},\underline{E}^{\bullet})$.

Proposion1.3 Let the fuzzy subspace U is induced by a fuzzy subset A of X, then $(U,\underline{E}^{+},\underline{E}^{\bullet})$ is a fuzzy subring iff (i) $(A_0,F^{+},F^{\bullet})$ is an ordinary ring, (ii) $f^{+}_{xy}(A(x),A(y)) = A(xF^{+}y)$, $f^{\bullet}_{xy}(A(x),A(y)) = A(xF^{\bullet}y)$, for all $A(x) \neq 0$ and $A(x) \neq 0$.

Proposion1.4 (i) Let $((X,I),\underline{E}^{+},\underline{E}^{\bullet})$ be a uniform fuzzy ring and let the comembership function f^{+} and $f^{\bullet}$ have the t-norm property and $f^{+} = f^{\bullet} = f$. Then every subset A of X, which induces fuzzy subring, is a classical fuzzy subring of the ring $(X,F^{+},F^{\bullet})$.

(ii) If $(Y,F^{+},F^{\bullet})$ is an ordinary subring of the ring $(X,F^{+},F^{\bullet})$, then every fuzzy subset, for which $A_0 = \{x \in X, A(x) \neq 0\} = Y$, induces fuzzy subrings of ring $((X,I),\underline{G}^{+},\underline{G}^{\bullet})$, such that $\underline{G}^{+} = (G^{+},g^{+}_{xy})$, $\underline{G}^{\bullet} = (G^{\bullet},g^{\bullet}_{xy})$, where $G^{+} = F^{+}$, $G^{\bullet} = F^{\bullet}$, and $g^{+}_{xy}(r,s)$, $g^{\bullet}_{xy}(r,s)$ are suitable comembership functions.

Proposion1.5 Every classical fuzzy subring A of ring $(X,F^{+},F^{\bullet})$ induces fuzzy subrings relative to some fuzzy ring $((X,I),\underline{G}^{+},\underline{G}^{\bullet})$.

2. Fuzzy Homomorphism

Definition2.1 A fuzzy homomorphism from a fuzzy ring $((X,I),\underline{E}^{+},\underline{E}^{\bullet})$ to a fuzzy ring $((Y,I),\underline{G}^{+},\underline{G}^{\bullet})$ is a fuzzy function, with onto comembership function

$$\underline{\theta} = (\theta,\theta_x) : (X,I) \to (Y,I)$$

such that

$$\theta((x_1,I)E^+(x_2,I)) = (\theta(x_1,I))G^+(\theta(x_2,I)),$$
$$\theta((x_1,I)E^\bullet(x_2,I)) = (\theta(x_1,I))G^\bullet(\theta(x_2,I)).$$

Theorem2.1 $((X,I),E^+,E^\bullet)$ and $((Y,I),G^+,G^\bullet)$ are homomorphic fuzzyrings iff the ordinary rings $(X,F^+,F^\bullet)$ and $(Y,G^+,G^\bullet)$ are homomorphic rings.
The proof is directly obtained from the ….

Theorem2.2 $((X,I),E^+,E^\bullet) \cong (X,F^+,F^\bullet)$.

The terms fuzzy monomorphism, epimorphism, isomorphism, endomorphism, and automorphism are defined in an obvious way(as in the ordinary case). Two fuzzy rings are said to be fuzzy isomorphic if there exists a fuzzy isomorphism beteen them.

Definition2.2 Let $U = \{(x,u_x); x \in U_0\}$ and $V = \{(x,v_x); x \in V_0\}$ be fuzzy subrings of $((X,I),E^+,E^\bullet)$ and $((Y,I),G^+,G^\bullet)$, respectively. A homomorphism from U to V is the fuzzy function $\theta = (\theta,\theta_x):(X,I) \to (Y,I)$ which satisfies:

(i) $\theta(x,u_x) = (\theta(x),\theta_x(u_x)) \in V$, i.e. $\theta_x(u_x) = v_{\theta(x)}$,

(ii) $\theta((x_1,u_{x_1})E^+(x_2,u_{x_2})) = (\theta(x_1,u_{x_1}))G^+(\theta(x_2,u_{x_2}))$,
$\theta((x_1,u_{x_1})E^\bullet(x_2,u_{x_2})) = (\theta(x_1,u_{x_1}))G^\bullet(\theta(x_2,u_{x_2}))$.

If θ is one-to-one, then we call θ an isomorphism.

Theorem2.3 The situation is same as above Definition2.2, then θ is a homomorphism iff

(i) $\theta_x(u_x) = v_x$,

(ii) $\theta: U_0 \to V_0$ is a homomorphism between the ordinary subrings $(U_0,F^+,F^\bullet)$ and $(V_0,G^+,G^\bullet)$.

(iii) $\theta_{xF^+y}(f^+_{xy}(u_x,u_y)) = g^+_{\vartheta(x)G^+\theta(y)}(v_{\theta(x)},v_{\theta(y)})$,
$\theta_{xF^\bullet y}(f^\bullet_{xy}(u_x,u_y)) = g^\bullet_{\vartheta(x)G^\bullet\theta(y)}(v_{\theta(x)},v_{\theta(y)})$.

Proof is easy and omitted.

Theorem2.4 Let $\theta = (\theta,\theta_x)$ be a fuzzy homomorphism between fuzzy ring $((X,I),E^+,E^\bullet)$ and $((Y,I),G^+,G^\bullet)$, let $U = \{(x,u_x); x \in U_0\}$ be a fuzzy subring of $((X,I),E^+,E^\bullet)$. If for all $\theta(x) = \theta(x')$, $\theta_x(u_x) = \theta_{x'}(u_{x'})$ holds, then

$$\theta(u) = \{(\theta(x),\theta_x(u_x); x \in U_0\}$$

is a fuzzy subring of $((Y,I),G^+,G^\bullet)$.

Proof. Obviously, $\theta(U_0)$ is a ordinary subring of $((Y,I),G^+,G^\bullet)$. For any $x,y \in X$,

$$g^+_{\vartheta(x)\theta(y)}(\theta_x(u_x),\theta_y(u_y)) = \theta_{xF^+y}(f^+_{xy}(u_x,u_y)) = \theta_{xF^+y}(u_{xF^+y}),$$
$$g^\bullet_{\vartheta(x)\theta(y)}(\theta_x(u_x),\theta_y(u_y)) = \theta_{xF^\bullet y}(f^\bullet_{xy}(u_x,u_y)) = \theta_{xF^\bullet y}(u_{xF^\bullet y}).$$

By Theorem2.2. $\theta(U)$ is a fuzzy subring of $((Y,I),G^+,G^\bullet)$. □

Definition2.3 Let A be a fuzzy subset, which induces fuzzy subrings of $((X,I),E^+,E^\bullet)$, and B a fuzzy subset, which induces fuzzy subrings of $((Y,I),G^+,G^\bullet)$. If ϑ is a fuzzy homomorphism between the fuzzy subring $H_0(A)$

of $((X,I),\underline{F}^+,\underline{F}^\bullet)$ and $H_0(B)$ of $((Y,I),\underline{G}^+,\underline{G}^\bullet)$, then we call $\underline{\vartheta}$ a fuzzy homomorphism between A and B.

Theorem2.5 If $\underline{\theta}$ is a fuzzy homomorphism between A and B, then $\overline{H}(A)$ is homomorphic to $\overline{H}(B)$ and $\underline{H}(A)$ is homomorphic to $\underline{H}(B)$ under the homomorphism $\underline{\theta}$.

Proof. If $\underline{\theta}$ is a fuzzy homomorphism between A and B, then

$$\underline{\theta}(x,\{0,A(x)\}) = (\theta(x),\{0,B(\theta(x))\}),$$

thus $\theta_x(\{0,A(x)\}) = \{0,B(\theta(x))\}$, and then $\theta_x(A(x)) = B(\theta(x))$, thus

$$\theta_x([0,A(x)]) = [0,B(\theta(x))] \text{ and } \underline{\theta}(x,[0,A(x)]) = (\theta(x),[0,B(\theta(x))]).$$

This means $\underline{\theta}$ is a homomorphism between $\underline{H}(A)$ and $\underline{H}(B)$. Another case is similar. □

Using the results above in the fuzzy subsets, which induce fuzzy subrings, we have the following special forms:

Theorem2.6 The fuzzy function $\underline{\theta} = (\theta,\theta_x): X \to Y$ is a fuzzy homomorphism between the fuzzy subset A, which induces fuzzy subrings of $((X,I),\underline{F}^+,\underline{F}^\bullet)$, and the fuzzy subset B, which induces fuzzy subrings of $((Y,I),\underline{G}^+,\underline{G}^\bullet)$ iff:

(i) $\theta: A_0 \to B_0$ is a crisp ring homomorphism of $(A_0,F^+,F^\bullet)$ and $(B_0,G^+,G^\bullet)$.

(ii) $\theta_{xF^+y}(f^+_{xy}(A(x),A(y)) = g^+_{\vartheta(x)G^+\theta(y)}(B(\theta(x)),B(\theta(y))$,

$\theta_{xF^\bullet y}(f^\bullet_{xy}(A(x),A(y)) = g^\bullet_{\vartheta(x)G^\bullet\theta(y)}(B(\theta(x)),B(\theta(y))$.

Theorem2.7 Let $\underline{\theta} = (\theta,\theta_x): ((X,I),\underline{F}^+,\underline{F}^\bullet) \to ((Y,I),\underline{G}^+,\underline{G}^\bullet)$ be a fuzzy homomorphism, if the fuzzy subset A of X induces fuzzy subrings of $((X,I),\underline{F}^+,\underline{F}^\bullet)$, then $\underline{F}(A)$ of (Y,I) induces fuzzy subrings of $((Y,I),\underline{G}^+,\underline{G}^\bullet)$ if $\theta_x(A(x)) = \theta_{x'}(A(x'))$ for all $\theta(x) = \vartheta(x')$.

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