The Fuzzy Ore Subrings

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Abstract

In [2], K. Asano have prove that a ring R can be extended to a fraction ring if and only if R is an Ore ring. In this paper, we give the fuzzy Ore conditions and obtain the fuzzy fractional ring of a fuzzy Ore ring and the fuzzy subfield of the commutative domain.

Keywords: Fuzzy subring, Ore ring, Fuzzy subfield

In classical algebra ring theory, the fractional ring of an Ore ring can be applied to many studies of mathematical structures. Recently, many results about fuzzy subrings and fuzzy subfields have been achieved. In this paper, the fuzzy subrings satisfying the fuzzy Ore conditions of an Ore ring R can be extended to a fuzzy fractional ring of the fractional ring of R. So that we can study the fuzzy subfields of the commutative domain.

Definition 1 Let Q be an extension ring of ring R. Q is called the fractional ring of R if Q satisfies the following conditions:

1) Each regular element w of R (w is called a regular element if $wx\neq 0$ and $yw\neq 0$, for any $x\neq 0, y\neq 0, x, y\in R$) has an inverse in R;

2) For any $x\in R$, there exist a, $b\in$R such that $x=a^{-1}b$.

Definition 2 Let R be a ring and M be the set of all regular elements of R. $M\neq\varnothing$. R is called an Ore ring if for any $a\in M$, $b\in R$, there exist $c\in M$ and $d\in R$ such that $da=cb$.

Theorem 1 ([2]) Let R be a ring. R has a fractional ring if and only if R is an Ore ring.

If ring R has fractional ring Q, Q can be built as following:

We give an equivalence relation $\sim$ in $R\times M$ (M is the regular set of R) $(a,b)\sim(c,d)$ if and only if for x, $y\in M$ if $xb=yd$ then $xa=yd$. Let $Q=R\times M/\sim$. For all (a,b),(c,d)$\in Q$, let $x,y\in M$ such that $xb=yd$ and $w\in M$, $z\in R$ such that $wa=zd$. We define

$$(a,b)+(c,d)=(xa+yc,yd),\ (a,b)(c,d)=(zc,wb).$$

By [2], [3], we know that Q is a fractional ring of R.

Definition 3 Let R be a ring and A is a fuzzy set of R. A is called a fuzzy subring of R if for any x, $y\in R$

1) $A(x-y)\geq\min\{A(x),A(y)\}$,

2) $A(xy)\geq\min\{A(x),A(y)\}$.

Definition 4 Let R be a Ore ring and M be the regular set of R. A fuzzy subring A of R is called a fuzzy Ore subring if A satisfies the following condition.

1) For any $m\in M$, for any $r\in R$, $A(r)\geq\min\{A(mr),A(m)\}$, $A(r)\geq\min\{A(mr),A(m)\}$.

2) For any $m\in M$, $r\in R$, there exist $w\in M$, $c\in R$ such that $cm=wr$ and $A(w)\geq A(m)$.

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Definition 5 Let R be a ring which has fractional ring Q and A is a fuzzy set of R. We define A/A a fuzzy set of Q as following: for any $(a,w)\in Q$,

$$A/A((a,w))=\sup_{(a,w)\sim(c,d)} \min\{A(c),A(d)\}.$$

In the following we always suppose that ring R is a ring with unit 1 and for any fuzzy Ore subring A, $A(1)\geq A(x)$, for any $x\in R$. The following theorem means that the fuzzy set A/A is an extended fuzzy set of fuzzy subring A.

Theorem 2 Let R be a Ore ring and Q is the fractional ring of R. A is a fuzzy Ore subring of R and M is the regular set of R. Then for any $r\in R$ and any $m\in M$

$$A/A(mr,m)=A(r).$$

Proof For any x, $y\in M$, if $xm=y1$, then $xmr=yr$. By the definition of the equivalence relation ~, we know that $(mr,m)\sim(r,1)$. For any $(mr,m)\sim(c,d)$, we have $(c,d)\sim(r,1)$. Since $1d=01$, hence $1c=dr=c$ and $(c,d)=(dr,d)$.

$$A/A((mr,m))=\sup_{(mr,m)\sim(c,d)}\min\{A(c),A(d)\}$$
$$\geq\min\{A(1r),A(1)\}=A(r).$$
$$A/A((mr,m))=\sup_{(mr,m)\sim(c,d)}\min\{A(c),A(d)\}$$
$$=\sup_{(mr,m)\sim(dr,d)}\min\{A(dr),A(d)\}.$$

Since A is a fuzzy Ore subring of R, hence

$$A(r)\geq\min\{A(dr),A(d)\}$$

and

$$A(r)\geq\sup_{(mr,m)\sim(dr,d)}\min\{A(dr),A(d)\}=A/A((mr,m)).\square$$

Theorem 3 Let R be an Ore ring and Q is its fractional ring. A is a fuzzy Ore subring of R. Then A/A is a fuzzy subring of Q.

Proof Let M be the regular set of R. For all $(a,b),(c,d)\in Q$, suppose $(a,b)(c,d)=(xc,yb)$, $(a,b)\sim(a',b')$, $(c,d)\sim(c',d')$ and $(a',b')(c',d')=(zc',wb')$. Since $d'\in M$ and the definition of fuzzy Ore subring, hence there exists $w\in M$, $z\in R$ such that $wa'=zd'$ and

$$A(w)\geq A(d'). \tag{1}$$

$$A(z)\geq\min\{A(zd'),A(d')\}=\min\{A(wa'),A(d')\}\geq\min\{A(w),A(a'),A(d')\}. \tag{2}$$

$$A/A((a,b)(c,d)) =A/A(xc,yb)$$
$$=\sup_{(xc,yb)\sim(zc',wb')}\min \{A(zc'),A(wb')$$
$$\geq\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')}\min \{A(z),A(c'),A(w),A(b')\}.$$

By (2) we have

$$A/A((a,b)(c,d))\geq\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')} \min \{A(w),A(c'),A(a'),A(d'),A(b')\}.$$

By (1) we have

$$A/A((a,b)(c,d))\geq\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')} \min \{A(c'),A(a'),A(d'),A(b')\}$$
$$=\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')}\min\{\min\{A(c'),A(a')\},\min\{A(d'),A(b')\}\}. \tag{3}$$

For any integer $n>0$, there exist $(k,l),(g,f)\in Q$ such that $(a,b)\sim(k,l),(c,d)\sim(g,f)$ and

$$\min\{A(k),A(l)\}\geq\sup_{(a,b)\sim(a',b')}\min\{A(a'),A(b')\}-1/n,$$
$$\min\{A(g),A(f)\}\geq\sup_{(c,d)\sim(c',d')}\min \{A(c'),A(d')\}-1/n.$$

$$\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')}\min\{\min\{A(c'),A(a')\},\min\{A(d'),A(b')\} \tag{4}$$
$$\geq\min\{\min\{A(k),A(l)\}, \min\{A(g),A(f)\}\}$$
$$\geq\min\{\sup_{(a,b)\sim(a',b')}\min\{A(a'),A(b')\}-1/n, \sup_{(c,d)\sim(c',d')}\min\{A(c'),A(d')\}-1/n\}\}.$$

When $n \to \infty$, we have

$$A/A((a,b)(c,d)) \geq \min\{\sup_{(a,b)\sim(a',b')}\min\{A(a'),A(b')\}, \sup_{(c,d)\sim(c',d')}\min\{A(c'),A(d')\}\}. \quad (5)$$
$$=\min\{A/A((a,b)),A/A((c,d))\}.$$

For any $(a,b),(c,d)\in Q$, suppose $(a,b)\sim(a',b')$, $(c,d)\sim(c',d')$. Since A is a fuzzy Ore subring, hence for $d'\in M$ and $b'\in M\subseteq R$, there exist $x\in M$, $y\in R$ such that $yd'=xb'$ and $A(x)\geq A(d')$,

$$A(y)\geq\min\{A(yd'),A(d')\}=\min\{A(xb'),A(d')\}\geq\min\{A(x),A(b'),A(d')\}.$$

By [2] we know that $y\in M$. $(a',b')-(c',d')=(xa'-xc',yd')$.

$$A/A((a,d)-(c,d)) =\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')}\min\{A(xa'-xc'),A(yd')\}$$
$$\geq\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')}\min\{A(xa'),A(xc'),A(y),A(d')\}$$
$$\geq\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')}\min\{A(xa'),A(xc'),A(x),A(b'),A(d')\}$$
$$\geq\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')}\min\{A(x),A(a'),A(c'),A(b'),A(d')\}.$$

Since $A(x)\geq A(d')$, hence

$$\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')}\min\{A(x),A(a'),A(c'),A(b'),A(d')\}$$
$$=\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')}\min\{A(a'),A(c'),A(b'),A(d')\}$$
$$=\sup_{(a,b)\sim(a',b'),(c,d)\sim(c',d')}\min\{\min\{A(a'),A(b')\},\min\{A(c'),A(d')\}\}$$
$$\geq\min\{\sup_{(a,b)\sim(a',b')}\min\{A(a'),A(b')\},\sup_{(c,d)\sim(c',d')}\min\{A(c'),A(d')\}\} \text{ (refer to (4),(5))}$$
$$=\min\{A/A(a',b'),A/A(c'd')\}. \square$$

Theorem 4 Let F be the fractional field of the commutative domain R. A is a fuzzy Ore subring of R. Then A/A is a fuzzy subfield of F.

Proof For any $(a,b)\in F$, a is no zero, (b,a) is the inverse of (a,b). It is obvious that if $(a,b)\sim(d,c)$, then $(b,a)\sim(c,d)$.

$$A/A((a,b)) =\sup_{(a,b)\sim(d,c)}\min\{A(c),A(d)\}$$
$$=\sup_{(b,a)\sim(c,d)}\min\{A(c),A(d)\}$$
$$=A/A((b,a)).$$

Therefore the fuzzy subring A/A is a fuzzy subfield. □

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