The Expression of Wavelets in a New Norm Space And Signal Processing

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Abstract. This paper illustrates the method of constructing symmtric wavelets with orthogonal property. The new inner product defined in scaling function space can be partly expressed as matrix form. Further, the application of symmetric wavelets in signal processing is proposed.

1. INTRODUCTION

It is well known that there are no orthogonal wavelet which is symmetric [2]. [3] proposes significant wavelets with convolution-type othorgonality conditions. These symmetric scaling and wavelet functions can obtain the orthogonality based a new definition of convolution-type orthogonality.. This paper attempts to illustrate symmetric wavelets with orthogonal property based a new norm space. In the space, the inner product can be express by a matrix, gene-matrix. By using the biorthogonal relation[1] , we realize the decomposition and reconstruction. Further, we show that the symmetric wavelets is a better selection in the signal processing.

Section 2 gives a $\perp$-inner product and $\perp$-norm concept and discusses matrix expression in the compactly supported situation. Section 3 states a wavelet and scaling functions. The last section proposes the future application of this symmetric wavelets .

2. NEW NORM IN SCALING FUNCTION SPACE

Let $V_1 = \{\sum_k c_k\phi(\cdot - k), c_k : real, \sum_k c_{0,k}^2 < \infty\}$ be a space spanned by the shifted scaling functions $\phi(\cdot - k)$, . Let $f(x) \in V_0$, we can find a mapping $\top$: $V_0 \to V_0^{\perp}$ such that $\top(f(x)) = f_{\perp}(x)$ and $f_{\perp}(x) = \sum_k c_k^{\perp}\phi_{\perp}(2x - k)$, where

$$c_k^{\perp} = \sum_{m=-M}^{M} p_{m,k}c_{k-m} \qquad (1)$$

and $p_{m,k} \in R,\ \ -M \le m \le M,\ \ k, M, m \in Z$. If let $g(x) \in V_1$, then we can give the $\perp$-operation defined as:

$$< f(x), g(x) >_{\perp} = < f_{\perp}(x), g(x) > = \int f_{\perp}(x)g(x)dx \qquad (2)$$

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On the other hand, $c_k^\perp$ can also be written as a matrix form

Further, we have

$$(c_n^\perp, \cdots, c_{-n}^\perp) = (c_{n+M}, \cdots, c_{-n-M}) \cdot P^\perp \tag{3}$$

where $P^\perp = (p_{-n}, p_{-n+1}, \cdots, p_n)$ is a matrix with $2(M+n)+1$ rows and $2n+1$ columns and p_i is column vectors with $2n$ zeros at least. If we impose $< \phi(\cdot - k), \phi(\cdot) >_\perp = \delta_{k0}$, then

$$\begin{aligned} < f, g >_\perp &= \sum_k \sum_l c_k^\perp d_l \int \sqrt{2}\phi_\perp(2x-k)\sqrt{2}\phi(2x-l)dx \\ &= \sum_l d_l^\perp c_l = \sum_n c_n^\perp d_n = R_{N+1} + M_N \end{aligned}$$

where $R_{N+1} = \sum_{n=-N-1}^{-\infty} c_n^\perp d_n + \sum_{n=N+1}^{\infty} c_n^\perp d_n$ is called the remains of $< f, g >_\perp$ and $M_N = \sum_{n=-N}^{N} c_n^\perp d_n$ the main value of $< f, g >_\perp$In fact, $\perp$-operation is really the inner product under the following assumptions.

H1: In $P^\perp$, $\bigcup_{-M \le m \le M} \bigcup_{k \in Z} p_{m,k} = \{h_1, h_2, \cdots, h_l \ \ h_i \in Z \ \ (1 \le i \le l)\}$

H2: In (6), there exists a large enough N such that $M_N \ge 0$.

Matrix $P^\perp$ in3 can be transformed into $P_0^\perp$, the details are as follows:

$$(c_{-N}^\perp, \cdots c_N^\perp) \begin{pmatrix} c_{-N} \\ \vdots \\ c_N \end{pmatrix} = (c_{N+M}, \cdots c_{-N-M}) P^\perp \begin{pmatrix} c_{-N} \\ \vdots \\ c_N \end{pmatrix} = (c_{-N}, \cdots c_N) P_0^\perp \begin{pmatrix} c_{-N} \\ \vdots \\ c_N \end{pmatrix}$$

where we call $(2N+1) \times (2N+1)$ matrix $P_0^\perp = (p_{-N}, \cdots, p_N)$ gene-matrix of $< \cdot, \cdot >_\perp$ and $p_i \ \ (-N \le i \le N)$ is $2N+1$ column vector.

In compactly supported situation, the $|| \cdot ||_\perp$ and $< \cdot, \cdot >_\perp$ depend on matrix $P_0^\perp$. If $P_0^\perp$ is selected as requirement (H1&H2), then $|| \cdot ||_\perp$ and $< \cdot, \cdot >_\perp$ is norm and inner product, and we call $\perp$-norm and $\perp$-inner product. Next, we will discuss the scaling and wavelet function based on $\perp$-norm and $\perp$- inner product.

3. WAVELETS IN NEW NORM SPACE

That the scaling function are orthogonal in $\perp$-inner product means:

$$< \phi(\cdot - k), \phi(\cdot) >_\perp = < \phi_\perp(\cdot - k), \phi(\cdot) > = \int \phi_\perp(x-k)\phi(x)dx = \delta_{k0} \tag{4}$$

where δ_{k0} denotes the Kronecker's delta symbol. $\perp$−operation is defined by 'traditional' norm and inner product, we mean $< f, g >_\perp = < f_\perp, g >$. Based on 'traditional' norm and inner product, the $\perp$-norm scaling and wavelet function can be dealt with biorthogonal situation. By the similar analysis method as biorthogonal wavelet, we can use the biorthogonal condition directly that (10) is equivalent to $\perp$-norm orthogonal condition

$$A(\zeta)\overline{u(\zeta)} + A(\zeta+\pi)\overline{u(\zeta+\pi)} = 1 \quad a.e. \tag{5}$$

where $A(\zeta) = \frac{1}{\sqrt{2}}\sum_{m=-M}^{M}\alpha_m^{\perp}e^{-in\zeta}$; $u(\zeta) = \frac{1}{\sqrt{2}}\sum_{m=-M}^{M}\alpha_m e^{-in\zeta}$. As [2]. we can deduce a useful result

$$\sum_l\left(\sum_{m=-M}^{M} p_{m,l}\alpha_{l+2k-m}\right)\alpha_l = \delta_{k0}, \quad k \in Z \tag{6}$$

which 6 can determine the coefficient as [3]. Further we have the orthogonal condition of wavelet correspond to scaling function is as the following:

$$v(\zeta)\overline{A(\zeta)} + v(\zeta+\pi)\overline{A(\zeta+\pi)} = 0$$

where $A(\zeta) = \frac{1}{\sqrt{2}}\sum_n \alpha_n^{\perp}e^{-in\zeta}$; $v(\zeta) = \frac{1}{\sqrt{2}}\sum_n \beta_n e^{-in\zeta}$. By the similar way of biorthogonal wavelets, coefficient β_n satisfies $\beta_n = (-1)^{n-1}\alpha_{1-n}^{\perp}$. Based on the concept of $\perp$-inner product, we certainly realize the decomposition and reconstruction formulas for signals as biorthogonal wavelets.

4. SIGNAL PROCESSING GAIN

The matched filter set used is the FFT which is matched to a set of equally spaced constant amplitude , constant frequency sinusoids. However if the FFT input has a center frequency not equal to one of the FFT filters of possesses amplitude or phase modulation, none the FFT filters is matched to the signal and processing loss relative to matched condition results. A wavelet-based method[7] is presented for recovering processing gain (PG)[6] lost in the fast Fourier transform (FFT) output due to weighting or nonbin center input frequency. The results illustrate that the FFT-DWT should be used with the FFT rather than as a replacement for the FFT. But the selection of the wavelets (Dbn[8]) may not be the optimum wavelet. Here we tried different series of wavelets such as Daubechies wavelets, Biorthogonal wavelets[1] and Coiflets wavelets, etc.. At last, we have found a better wavelet than [7] and show that the method is effective for the optimal problem.

The one-level DWT for signal is given by $d^1(k) = \sum_n g(n-2k)X(n)$, where $X(n)$ is the FFT output in the nth bin and $g(n)$ are the wavelet filter coefficients. By substituting the expression of the FFT output into the DWT expression. The power of the wavelet coefficient corresponding to the target is $\max_k(|d_s^1(k)|^2) = \max_k(|\sum_n \mathbf{g(n-2k)}X_s(k)|^2)$. The DWT noise output is $d_N^1(k) = \sum_n g(n-2k)X_N(k)$.

The biorthogonal wavelets have linear phase. Thus it performs best in recovering the PG loss. The selection of wavelets is partially dependent on the phase of input FFT signal. Since the symmetric wavelet has fine linear phase, we should consider the symmetric coefficients of wavelets to decomposition in optimal problem. The authors will consider further application of above wavelets in signal processing.

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