α, β Operators On Quasi-Boolean Chain

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Abstract:At first,we propose the α operator and β operator on quasi-Boolean chain.Next investigate their properties and apply them to solve inqualities in one unknown.
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1 Basic Definitions

Definition1.1. A partially ordered set(or briefly,a poset) is a pair (P, $\leq$) where P is a non-empty set and $\leq$ is a binary relation on P satisfying for $x, y, z \in P$

1. $x \leq x$ (reflexive law)
2. $x \leq y, y \leq z \Longrightarrow x \leq z$ (transitive law)
3. $x \leq y, y \leq x \Longrightarrow x = y$ (anti-symmetric law)

Where no confusion is likely to arise,it is customary to also use the symbol P to denote $(P, \leq)$. Also,if $x \leq y$ but $x \neq y$ then we write $x < y$

Definition 1.2.If P is a poset then a non-empty subset C of P is called a chain in P if and only if any two elements in C are comparable.If P=C then P is a chain.(chains are also refered to as linearly or totally ordered posets)Two members a,b of poset are comparable if $a \leq b$ or $b \leq a$.

Let P be a poset.There is at most one element b in P with the property that $x \leq b$ for all $x \in P$. This element,if it exists,is called the greatest element or the unit of P. Dually,a least element b(or zero)by the property that $b \leq x$ for $x \in P$. The zero and unit of P are denoted,when they exist,by 0_p and 1_p (or simply by 0,1).

Definition 1.3. A lattice is a poset in which x+y and x.y exist for any $x, y \in L$

In a lattice L,the following two statements are equivalent:

1. $x(y+z) = xy + xz$ for all x,y,z in L (1.1)

2. $u + vw = (u+v)(u+w)$ for all u,v,w in L (1.2)

Definitin 1.4. A lattice L is distributive if it satisfies one(and hence both)of (1.1) and (1.2)

Definition 1.5. A quasi-Boolean algebra is an algebra of the form $(L, (+, ., -))$ where $+$ and $.$ are binary operations, $-$ is a unary operation, satisfying

1. $(L, (+, .))$ is a distributive lattice with 0,1

2. $\overline{x+y} = \overline{x}.\overline{y}; \overline{xy} = \overline{x} + \overline{y}$. for all $x, y \in L$

3. $\overline{\overline{x}} = x$. for each $x \in L$

In this paper,we assume that L is a chain with a kernel element denoted by e, which is defined by the property: $e = \overline{e}$

Example 1: $L = [0,1], a + b = \min(a,b), \overline{a} = 1 - a$, then $(L, (+, ., -))$ is a quasi-Boolean algebra,and L is a chain with a kernel element e ,where e=0.5

Example 2. $L = \{a_1, a_2, a_3, a_4, a_5\}, \overline{a_1} = a_5, \overline{a_2} = a_4, a_3 = e$, L is a quasi-Boolean chain with a kernel element e shown as Fig.1

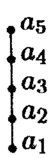


Fig.1

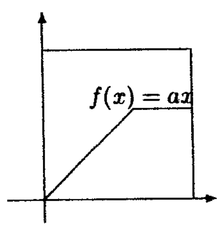
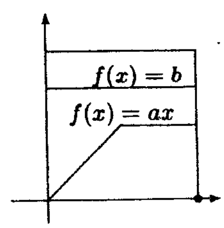


Fig.2



If $a \leq b$
Then $S = [0,1], a\alpha b = 1$
Fig.3

2 Operators on Quasi-Boolean Chain

Let $a, b \in C$, define

$$a\alpha b = \sum\{x | ax \leq b\}$$

Since C is a complete lattice,so $\sum\{x|ax \leq b\} \in C$

Theorem 2.1. $a\alpha b$ is the greatest element of the set $\{x|ax \leq b\}$

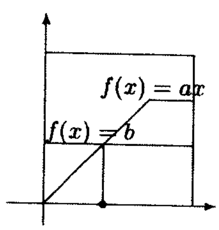
Proof. First, $a\alpha b \in \{x|ax \leq b\}$. In fact:

Let $X(a,b) = \{x|ax \leq b\}$. $\forall x_0 \in X(a,b)$, then $ax_0 \leq b$. Now $\sum\limits_{x_0 \in X(a,b)} ax_0 = a \sum\limits_{x_0 \in X(a,b)} x_0 = a(a\alpha b)$. Again, $\sum\limits_{x_0 \in X(a,b)} ax_0 \leq b$. So $a(a\alpha b) \leq b$. Hence $a\alpha b \in X(a,b)$.

Next.Suppose $u \in X(a,b)$, then it is easy to see that $u \leq \sum\limits_{x_0 \in X(a,b)} x_0 = a\alpha b$. So The proof. is completed

Function $f(x) = ax$ as shown in Fig.2

Consider the solution sets of the inequality $ax \leq b$: (1) $a \leq b$: The solution set shown as Fig.3;(2) $a > b$: the solution set shown as Fig.4



If $a > b$
Then $S = [0, b], a\alpha b = b$
Fig.4

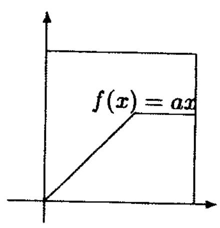


Fig.5

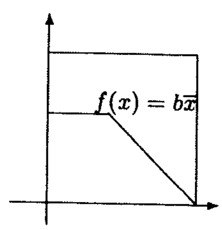


Fig.6

Hence we can define a operator α by the following

$$a\alpha b = \begin{cases} 1, & a \leq b \\ b, & a > b \end{cases}$$

Let $a, b \in C$. define

$$a\beta b = \sum\{y | ay \leq b\overline{x}\}$$

Since C is a complete lattice,so $\sum\{y|ay \leq b\overline{y}\} \in C$

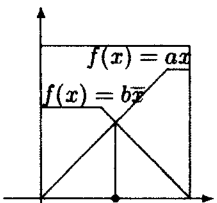
Throrem2.2. $a\beta b$ is the greatest element of the set $\{y|ay \leq b\overline{y}\}$

Proof.First, $a\beta b \in \{y|ay \le b\bar{y}\}$ Indeed

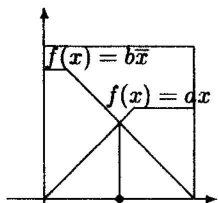
Let $Y(a,b) = \{y|ay \le b\bar{y}\}$. For each $p \in Y(a,b)$, then $ap \le b\bar{p}$; For any $q \in Y(a,b)$, so $aq \le b\bar{q}$. Case 1. $p \le q$: It is obviously that $ap \le aq \le b\bar{q}$. We can obtain that $ap \le \prod_{q\in Y(a,b)} b\bar{q}$, and than $ap \le b \prod_{q\in Y(a,b)} \bar{q}$. Again $\prod_{q\in Y(a,b)} \bar{q} = \overline{\sum_{q\in Y(a,b)} q} = \overline{a\beta b}$. So $ap \le b\overline{a\beta b}$. It is easy to see that $\sum_{p\in Y(a,b)} ap \le b\overline{a\beta b}$. Again $\sum_{p\in Y(a,b)} ap = a \sum_{p\in Y(a,b)} p = a(a\beta b)$. Hence $a(a\beta b) \le b\overline{a\beta b}$. Case2 $p > q$: It is similarly prove that $a(a\beta b) \le b\overline{a\beta b}$.

Next,Suppose $u \in \{y|ay \le b\bar{y}\}$ .It is easy to see that $u \le a\beta b$.

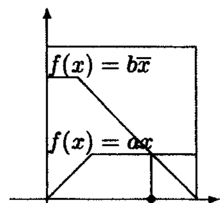
The function $f_1(x) = ax, f_2(x) = b\bar{x}$ shown as Fig.5,Fig.6 respectivly.The solution sets of inquality $ax \le b\bar{x}$ shown as Fig.7—Fig.12,respectivly



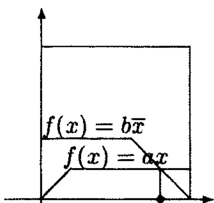
If $b \ge e, a \ge e, a \ge b$
Then $S = [0,e], a\beta b = e$
Fig.7



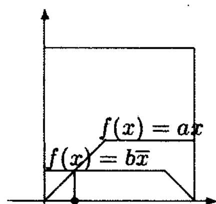
If $b \ge e, a \ge e, a < b$
Then $S = [0,e], a\beta b = e$
Fig.8



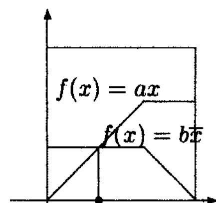
If $b \ge e, a < e$
Then $S = [0,\bar{a}], a\beta b = \bar{a}$
Fig.9



If $b < e, a \le b$
Then $S = [0,\bar{a}], a\beta b = \bar{a}$
Fig.10



If $b < e, a > b, a < e$
Then $S = [0,b], a\beta b = b$
Fig.11



If $b < e, a \ge e, a > b$
Then $S = [0,b], a\beta b = b$
Fig.12

We can define a operator by the following

$$a\beta b = \begin{cases} e, & b \ge e, a \ge e \\ \bar{a}, & b \ge e, a < e \\ \bar{a}, & b < e, a \le b \\ b, & b < e, a > b \end{cases}$$

It is easy to obtain the following results.

Theorem2.3. $\forall a, b \in C$

1. $a\alpha b \ge b$

2. If $a \leq b$, then $a\beta b \geq e$.

3. If $e \leq b$, then $a\beta b \geq e$.

Theovem2.4 $\forall a, b, c \in c$, then the follwing statements are equivalent:

1. $ace \leq b$

2. $\overline{a\alpha b} \leq c\alpha b$

3. $\overline{a\alpha b} \leq c\beta b$

4. $\overline{a\beta b} \leq c\alpha b$

5. $\overline{a\beta b} \leq c\beta b$

Proof. (1) $\Longrightarrow$ (2) If $ace \leq b$, then $a \leq b$ or $c \leq b$ or $e \leq b$. Case1 $a \leq b$: since $a\alpha b = 1$, so $\overline{a\alpha b} = 0$. It is obvionsly that $\overline{a\alpha b} \leq c\alpha b$; Case2 $c \leq b$: since $c\alpha b = 1$, hence $\overline{a\alpha b} \leq c\alpha b$; Case3 $e \leq b$: since $a\alpha b \geq b$ and $c\alpha b \geq b$, so $\overline{a\alpha b} \leq \bar{b} \leq c\alpha b$, thus $\overline{a\alpha b} \leq c\alpha b$.

(2) $\Longrightarrow$ (1) Suppose that $ace > b$. We have $a > b$, $c > b$, and $e > b$. By definition,then $a\alpha b = b$, and $c\alpha b = b$. Again $\overline{a\alpha b} = \bar{b}$ and $e > b$, so $\overline{a\alpha b} > c\alpha b$. A contradiction.

(1) $\Longrightarrow$ (3) If $ace \leq b$, then $a \leq b$ or $c \leq b$ or $e \leq b$. case 1 $a \leq b$: since $a\alpha b = 1$, so $\overline{a\alpha b} = 0$. It is easy to see that $\overline{a\alpha b} \leq c\beta b$; Case 2 $c \leq b$: If $b < e$, then $c\beta b = \bar{c}$. Again $a\alpha b \geq b$. So $\overline{a\alpha b} \leq \bar{b} \leq \bar{c} = c\beta b$. Hence $\overline{a\alpha b} \leq c\beta b$; If $b \geq e$, then Note that $a\alpha b \geq b$, so $\overline{a\alpha b} \leq \bar{b} < e \leq c\beta b$. Hence $\overline{a\alpha b} \leq c\beta b$; Case 3 $e \leq b$: Then $a\alpha b \geq b$ and $c\beta b \geq e$, so $\overline{a\alpha b} \leq \bar{b} \leq e \leq c\beta b$. We have $\overline{a\alpha b} \leq c\beta b$.

(3) $\Longrightarrow$ (1) Suppose that $ace > b$, we obtain that $a > b, c > b, e > b$. So $a\alpha b = b$ and $c\beta b = b$. Then $\overline{a\alpha b} = \bar{b} > b = c\beta b$. It follows that $\overline{a\alpha b} > c\beta b$. A contradiction.

(1) $\Longrightarrow$ (4) If $ace \leq b$, then $a \leq b$ or $c \leq b$ or $e \leq b$. Case 1 $a \leq b$: If $b < e$, then $a\beta b = \bar{a}$. So $\overline{a\beta b} = a \leq b \leq c\alpha b$; If $b \geq e$, then $a\beta b \geq e$. So $\overline{a\beta b} \leq e \leq b \leq c\alpha b$; Case 2 $c \leq b$: we obtain that $c\alpha b = 1$. So $\overline{a\beta b} \leq c\alpha b$; Case 3 $e \leq b$: Then $a\beta b \geq e$, so $\overline{a\beta b} \leq e < b \leq c\alpha b$. Hence $\overline{a\beta b} \leq c\alpha b$.

(4) $\Longrightarrow$ (1) Suppose that $ace > b$, then $a > b, c > b, e > b$. It is easy to see that $c\alpha b = b, a\beta b = b$. So $\overline{a\beta b} = \bar{b} > b = c\alpha b$. A contradiction.

(1) $\Longrightarrow$ (5) If $ace \leq b$, then $a \leq b$ or $c \leq b$ or $e \leq b$. Case 1 $a \leq b$: If $b < e$, then $a\beta b = \overline{a}$. So $\overline{a\beta b} = a \leq b \leq c\beta b$: If $b \geq e$, then $a\beta b \geq e$ and $c\beta b \geq e$. It follows that $\overline{a\beta b} \leq e \leq c\beta b$; Case 2 $c \leq b$: We have $c\beta b \geq e$. If $b < e$, then $c\beta b = \overline{c}$. It is obviously that $\overline{c\beta b} = c \leq b \leq c\beta b$. Hence $c\beta b = \overline{c} \geq \overline{b} \geq \overline{a\beta b}$; If $b \geq e$, then $a\beta b \geq e$ and $c\beta b \geq e$. Clearly $\overline{a\beta b} \leq c\beta b$.

(5) $\Longrightarrow$ (1) Suppose that $ace > b$, Then $a > b, c > b, e > b$ It follows that $a\beta b$ and $c\beta b = b$. It is obviously that $\overline{a\beta b} = \overline{b} > b = c\beta b$. A contradiction.

3 Inquality in One Unknown on Quasi-Boolean Chain

We have discussed the properties of "α" operater and "β" operater on quasi-Boolean chain,Now we apply it to solve Inqualities in one unknown.

Consider the following inquality.

$$a_1x + c_1\overline{x} \leq a_2x + b_2x\overline{x} + c_2\overline{x} \tag{3.1}$$

Now construct the inqualities

$$a_1x + c_1\overline{x} \leq a_2x \tag{3.11}$$

$$a_1x + c_1\overline{x} \leq b_2x\overline{x} \tag{3.12}$$

$$a_1x + c_1\overline{x} \leq c_2\overline{x} \tag{3.13}$$

Theorem3.1 The inquality (3.1) is consistent if and only if $a_1c_1e \leq a_2 + b_2 + c_2$, and

1. If $a_1c_1e \leq a_2$, then the set $[\overline{c_1\beta a_2}, a_1\alpha a_2]$ is the solution of the inquality (3.1)
2. If $a_1c_1e \leq b_2$, then the set $[\overline{c_1\beta b_2}, a_1\beta b_2]$ is the solution of the inquality(3.1)
3. If $a_1c_1e \leq c_2$, then the set $[\overline{c_1\alpha c_2}, a_1\beta c_2]$ is the solution of inquality(3.1)

Proof. First Inquality (3.11) is consistent if and only if $[0, a_1\alpha a_2]\bigcap[\overline{c_1\beta a_2}, 1] \neq \emptyset$. Again $\overline{c_1\beta a_2} \leq a_1\alpha a_2$ if and only if $a_1c_1e \leq b_2$ and inquality (3.12) is consistent if and only if $a_1c_1e \leq b_2$ and inquality (3.13) is consistent if and only if $a_1c_1e \leq c_2$. Note that inquality (3.1) is consistent if and only if there is at least a inquality among inquality (3.11),(3.12) and (3.13) is consistent and the solution set ofinquality (3.1) is the union of their solution set.So inquality (3.1) is cousistent if and only if $a_1c_1e \leq a_2 + b_2 + c_2$.

Next consider the following inquality

$$a_1x + b_1x\overline{x} + c_1\overline{x} \leq a_2x + b_2x\overline{x} + c_2\overline{x} \tag{3.2}$$

Construct two inqualities

$$(a_1 + b_1)x + c_1\overline{x} \leq a_2x + b_2x\overline{x} + c_2\overline{x} \tag{3.21}$$

$$a_1x + (b_1 + c_1)\overline{x} \leq a_2x + b_2x\overline{x} + c_2\overline{x} \tag{3.22}$$

It is obviously that

Therem 3.2. Inquality (3.2) is consistent if and only if $a_1c_1e \leq a_2 + b_2 + c_2$ and its solntion set is the union of the solution set of inqualities (3.21),(3.22)

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