Homomorphism of L-fuzzy topological groups*

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Abstract: Definitions such as L-fuzzy homomorphism and L-fuzzy open homomorphism between L-fuzzy topological groups are introduced. We describe their characters and prove that L-fuzzy topological group is $L - good\ extension$. Furthermore, the paper shows relationship between L-fuzzy homomorphism and general topological groups. Some characters of the relationship are discussed.

Key words: L-fuzzy topological groups, L-fuzzy homomorphism, L-fuzzy topological spaces

I Preliminary

The denotation of L stands for fuzzy lattice and L-fuzzy (LF) topological spaces are full level. P(L)[6] is a set which includes all the non-one prime units. The terms and detonations about LF topological spaces are talked in the paper[3]. But the paper[9] introduces the definition of remote domain.

The LF mapping in this paper is induced by some general mappings.

Let the mapping $f: X \to Y$ be a general mapping. The induced LF mapping is defined by

$$f: X^X \to Y^Y, \forall A \in L^X, \forall B \in L^Y, \forall x \in X, \forall y \in Y.$$

$$f_\bullet(A)(y) = \vee\{A(x) : f(x) = y\}, f_\bullet^{-i}(B)(x) = B(f(x)).$$

In the following part the signs of G, G_1, G_2 are general groups. AB and A^{-1} for $A, B \in L^G$ are defined as follows: $(AB)(x) = \bigvee_{x = x_1 x_2} A(x_1) \wedge B(x_2), A^{-1}(x) = A(x^{-1})$.

Proposition 1.1

(1) LF mapping $f_\bullet : L^G \times L^G \to L^G, A \times B \mapsto AB$ can be induced by the following general mapping

$$f: G \times G \to G, (x, y) \mapsto xy$$

(2) LF mapping $g_\bullet : L^G \to L^G, A \mapsto A^{-1}$ can be induced by the following general mapping

$$g: G \to G, x \mapsto x^{-1}$$

Proof: (1) $\forall z \in G, f_\bullet(A \times B)(z) = \vee\{(A \times B)(x, y) : f(x, y) = z\} = \vee\{A(x) \wedge B(y) : xy = z\}$

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$= (AB)(z)$

(2) $\forall y \in G, g_{\bullet}(A)(y) = \vee \{A(x) : g(x) = y\} = \vee \{A(x) : x^{-1} = y\} = \vee \{A(x) : x = y^{-1}\} = A(y^{-1}) = A^{-1}(y)$

Definition1.2 LF topological spaces is called LF topological groups, if the following conditions are satisfied:

(G1) LF mapping $f_{\bullet} : (L^G, \delta) \times (L^G, \delta) \to (L^G, \delta)$ in the remark 1.1 is LF continuous.

(G2) LF mapping $g_{\bullet} : (L^G, \delta) \to (L^G, \delta)$ in the remark 1.1 is LF continuous.

Note 1.3 Though the definition given in this paper differs from that in the paper [4] in the form, their essences are the same. We think that the definition in this paper seems to embody the essence of LF topological groups. Moreover, it is easier to distinguish general topological groups.

Proposition 1.4: Three remarks are right for $A, B \in L^G$ and $a, x \in G$,

(1) $(aA)(x) = A(a^{-1}x), (Aa)(x) = A(xa^{-1})$

(2) $(aA)' = aA', (Aa)' = A'a$

(3) $(A')^{-1} = (A^{-1})', (AB)^{-1} = B^{-1}A^{-1}, (A^{-1})^{-1} = A$

II LF homomorphism of LF topological groups

Definition2.1: Let (L^{G_1}, δ_1) and (L^{G_2}, δ_2) be LF topological groups. And

let $f_{\bullet} : (L^{G2}, \delta_1) \to (L^{G2}, \delta_2)$ be a LF mapping, and general mapping $f : G_1 \to G_2$ induced by $f_{\bullet}$ be group homomorphism.

(1) $f_{\bullet}$ is called LF homomorphism between LF topological groups (L^{G_1}, δ_1) and0 (L^{G_2}, δ_2) (LF homomorphism), if $f_{\bullet}$ is LF continuous.

(2) $f_{\bullet}$ is called LF open homomorphism between LF topological groups (L^{G_1}, δ_1) and (L^{G_2}, δ_2) (LF open homomorphism),if $f_{\bullet}$ is open mapping with LF continuity.

It is well known that it is at the point of unit of groups that general mapping is continuous or open in general topological groups. Now we will extend this result to LF topological groups. Let us first introduce LF open mapping at a point.

Definition 2.2

Let (L^X, δ) and (L^Y, σ) be LF topological spaces. LF mapping $f : (L^X, \delta) \to (L^Y, \sigma)$ is open at the point of $x_\lambda \in M(L^X)$,if there exists $V \in \eta((f(x))_\lambda)$ for $\forall U \in \eta(x_\lambda)$ such that $(f(U'))' \leq V$.

Proposition 2.3

Let (L^X,δ) and (L^Y,σ) be LF topological spaces. LF mapping $f:(L^X,\delta)\to(L^Y,\sigma)$ is open if and only if f is open at each point $x_\lambda \in M(L^X)$.

Proof: Necessity:

Because it is right that $U'\in\delta$ for $\forall U\in\eta(x_\lambda)$, W equals to $f(U')$.

From following equations $W'(f(x))=(f(U'))'(f(x))=(f(U')(f(x)))'=(\bigvee_{x\in f^{-1}(x)} U'(x))'$

$$=\bigwedge_{x\in f^{-1}(x)} U(x)\le U(x)\not\geq\lambda$$

W' belongs to $\eta((f(x))_\lambda)$. Let V equal to W', such that $(f(U'))'=W'\le V$.

Sufficiency: It is proved that $f(A)$ belongs to σ for every A of δ. It is equivalent that $(f(A))'$ belongs to σ'. In order to make the above result true, $(f(A))^{\prime -}$ is not bigger than $(f(A))'$. Therefore it is sufficient that y_a doesn't belong to $(f(A))'^{-}$ on the condition that y_a doesn't belong to $(f(A))'$ for every y_a of $M(L^X)$

Because $a\not\geq(f(A))'(y)=(f(A)(y))'=(\vee\{A(x):f(x)=y\})'=\wedge\{A'(x):f(x)=y\}$, there exists $x\in X$ and $f(x)=y$ such that $a\not\geq A'(x)$. So A' belongs to $\eta(x_a)$. From the supposition of remark there exists $W\in\eta((f(x))_a)=\eta(y_a)$ such that $(f(A))'\le W$. It show that y_a isn't a attached point of $(f(A))'$. Therefore it is concluded that y_a doesn't belong to $(f(A))'^{-}$

End of proof.

Lemma 2.4[2,4] (1) The following results are right for $A,B,C\in L^G$,

(1°) $AC\le BC$ and $CA\le CB$ if $A\le B$.

(2°) $(AB)C=A(BC)$.

(2) $f_\bullet(AB)$ equals to $f_\bullet(A)f_\bullet(B)$ for $A,B\in L^{G_1}$ if $f:G_1\to G_2$ is a group homomorphism. Here $f_\bullet:L^{G_1}\to L^{G_2}$ is a LF mapping induced by f.

Theory 2.5 Let (L^{G_1},δ_1) and (L^{G2},δ_2) be LF topological groups. And $f:G_1\to G_2$ is a group homomor phism. In addition e_i is unit of G_i ($i=1,2$). LF mapping $f_\bullet:(L^{G_1},\delta_1)\to(L^{G2},\delta_2)$ can be induced by f.

(1) $f_\bullet$ is continuous if and only if there exists $U\in\eta((e_1)_\lambda)$ for every λ of $M(L)$ and every U of

$\eta((e_2)_\lambda)$ such that $V \le (f_\bullet(U'))'$.

(2) $f_\bullet$ is open if and only if there exists $V \in \eta((e_2)_\lambda)$ for every λ of $M(L)$ and every U of $\eta((e_1)_\lambda)$ such that $(f_\bullet(U'))' \le V$.

Proof (1) Necessity: It is supposed that $f_\bullet$ is continuous. Because f is grouphomomorphism, $f(e_1)$ equals to e_2 for $\forall \lambda \in M(L), \forall V \in \eta((e_2)_\lambda) = \eta((f(e_1))_\lambda)$, $f_\bullet^{-1}(V) \in \eta((e_1)_\lambda)$.Let U eqial to $f_\bullet^{-1}(V)$. So $f_\bullet(U') = f_\bullet((f_\bullet^{-1}(V))') = f_\bullet(f_\bullet^{-1}(V')) \le V'$ Therefore $V \le (f_\bullet(U'))'$.

Sufficiency: $\forall B \in \delta_2'$. If we will prove that $x_\lambda \notin f_\bullet^{-1}(B) \Rightarrow x_\lambda \notin (f_\bullet^{-1}(B))^-$ for every x_λ of $M(L^{G_1})$, $f_\bullet^{-1}(B)$ belongs to δ_1'.Because $\lambda \not\le f_\bullet^{-1}(B)\,(x) = B(f(x)) = B(y)$, $B \in \eta(y_\lambda)$.Also we can conclude that $By^{-1} \in \eta((e_2)_\lambda)$ from conditions of $(By^{-1})(e_2) = B(e_2 y) = B(y)$ and $By^{-1} \in \delta_2'$ (the remark of the paper[4]). By the supposition of theory there exists P of $\eta((e_1)_\lambda)$ such that $By^{-1} \le (f_\bullet(P'))'$.

From $Q(x) = (px)(x) = p(xx^{-1}) = p(e_1)$ and $\lambda \not\le P(e_1)$,we know that $Q \in \eta(x_\lambda)$ (that Q belongs to δ_1' is the remark of the paper[4]). Because $f_\bullet(Q') = f_\bullet((Px)') = f_\bullet(P'x) = f_\bullet(P')f(x)$ (Lemma 2.4(2)) $= f_\bullet(P')y \le (By^{-1})'y = (B'y^{-1})y = B'(y^{-1}y)$ (Lemma 2.4(1)(2°)) $= B'e_2 = B'$, $B \le (f_\bullet(Q'))'$ .Therefore $f_\bullet^{-1}(B) \le f_\bullet^{-1}((f_\bullet(Q'))' = (f_\bullet^{-1}(f_\bullet(Q')))' \le Q'' = Q$. This shows that x_λ isn't a attached point of $f_\bullet^{-1}(B)$. In a word $x_\lambda \notin (f_\bullet^{-1}(B))^-$.

(2)Necessity: It can be proved according to definition 2.2 and lemma 2.3.

Sufficiency: It must be proved that $f_\bullet$ is open at the point of x_λ.We can conclude that $Ux^{-1} \in \eta((e_1)_\lambda)$ from $Ux^{-1} \in \delta_1'$ (remark of the paper[4]) and $(Ux^{-1})(e_1) = U(e_1 x) = U(x) \not\ge \lambda$.There exists V of $\eta((e_2)_\lambda)$ such that $(f_\bullet((Ux^{-1})'))' \le V$ .We need to notice that $Vf(x) = Vy \in \delta_2'$.

Because $(Vy)(y) = \eta((e_2)_\lambda) = V(e_2) \not\ge \lambda$, Vy belongs to $\eta(y_\lambda)$.In the following part we will prove that $(f_\bullet(U'))' \le W \overset{\Delta}{=} Vy$ .Because

$$(f_\bullet((Ux^{-1})'))' = (f_\bullet(U'x^{-1}))' = (f_\bullet(U')f(x^{-1}))' \text{ (Lemma 2.4(2))}$$

$$= (f_\bullet(U'))'f(x^{-1}) \text{ (Lemma 1.4(2))}$$

$$=(F_{\bullet}(U'))'(f(x))^{-1}=(f_{\bullet}(U'))'y^{-1},$$

we can get that $(f_{\bullet}(U'))'y^{-1}\leq V$ .According to Lemma 2.4(1)(1°) $(f_{\bullet}(U'))'\leq Vy=W$ From definition 2.2 $f_{\bullet}$ is open at the point of x_{λ}.

End of proof.

III Relationship between LF homomorphism and general homomorphism

Lemma 3.1[5] Let $(L^{G},w_{L}(\tau))$ be LF topology induced by general topological spaces (G,τ). (G,τ) is topological groups if and only if $(L^{G},w_{L}(\tau))$ is LF topological groups.

From Lemma 3.1 it can be proved that LF homomorphism is $L-good\ \ extension$.

Theory 3.2 Let $(L^{G_1},w_{L}(\tau_1))$ and $(L^{G_2},w_{L}(\tau_2))$ be LF topological groups induced by general topological groups (G_1,τ_1) and (G_2,τ_2),respectively. $f:(G_1,\tau_1)\rightarrow(G_2,\tau_2)$ is a general homomorphism if and only if LF mapping $f_{\bullet}:(L^{G_1},w_{L}(\tau_1))\rightarrow(L^{G_2},w_{L}(\tau_2))$ induced by f is a LF homomorphism.

Proof It is enough to prove that f is generally continuous if and only if LF is continuous. This is right because of theory 2.11.22 of the paper[3].

Lemma 3.3[5] If (L^{G},δ) is a LF topological group and LF topology is induced weakly, surply space $(G,[\delta])$ is a general topological group.

Theory 3.4; Let (L^{G_1},δ_1) and (L^{G_2},δ_2) be LF topological groups and δ_1 and δ_2 be induced weakly. $f_{\bullet}:(L^{G_1},\delta_1)\rightarrow(L^{G_2},\delta_2)$ is LF homomorphism if and only if general mapping $f:(G_1,[\delta_1])\rightarrow(G_2,[\delta_2])$ which induces $f_{\bullet}$ is a general homomorphism.

Proof: It is only to prove that f is continuous if and only if $f_{\bullet}$ is LF continuous.

Let $f_{\bullet}$ be continuous. If B belongs to $[\delta_2]$, $\chi_B\in\delta_2$ (χ_B is characteristic function of B). Therefore $f_{\bullet}^{-1}(\chi_B)\in\delta_1$ .It is easy to prove that $f_{\bullet}^{-1}(\chi_B)=\chi_{f^{-1}(B)}$ .Furthermore, $\chi_{f^{-1}(B)}\in\delta_1$ and $f^{-1}(B)\in[\delta_1]$ .This proves that f is continuous. On the contrary if f is continuous , $l_a(f_{\bullet}^{-1}(A))=f^{-1}(l_a(A))$ for every A of $[\delta_2]$ and every a of L .Because δ_2 is induced weakly, $l_a(A)\in[\delta_2]$ and $f^{-1}(l_a(A))\in[\delta_1]$ .Therefore $\chi_{l_a}(f_{\bullet}^{-1}(A))\in\delta_1$ .According to the paper[3], $f_{\bullet}^{-1}(A)\in\delta_1$. This shows that $f_{\bullet}$ is continuous.

End of proof.

Lemma 3.5 If (L^G,δ) be a LF topological group, $(G,l_L\delta))$ is a LF topological group.

Proof $\forall x,y\in G,\forall V\in N(xy^{-1})$ where $N(xy^{-1})$ stands for open domain system of xy^{-1} in the $(G,l_L(\delta))$. Because $\phi(\delta)=\{l_r(A):r\in P(L),A\in\delta\}$ is subbasis of $(G,l_L(\delta))$,there exists $r_i\in P(L)$ and $A_i\in\delta(i=1,2,....,n)$ such that $xy^{-1}\in\bigcap_{i=1}^{n}l_{r_i}(A_1)\subset V$. So $A_i(xy^{-1})\not\leq r_i$ for $\forall i$.

Furthermore, $A_i'\in\eta((xy^{-1})_{r_i'})$. Because (L^G,δ) is a LF topological group, there exists $Q_i\in\eta(x'_{r_i})$ and $R_i\in\eta(y'_{r_i})$ such that $A_i'\leq(Q_i'(R_i^{-1})')',i=1,2,.....,n$.(theory 1 of the paper[5]).In addition, $x\in l_{r_i}(Q_i')\in N(x),y\in l_{r_i}(R_i')\in N(y),i=1,2,....,n$ .Let $V_1=\bigcap_{i=1}^{n}l_{r_i}(Q_i'),V_2=\bigcap_{i=1}^{n}l_{r_i}(R_i')$. So $V_1\in N(x)$ and $V_2\in N(y)$ .Because $A_i'\leq(Q_i'(R_i^{-1})')',A_i\geq Q_i'(R_i^{-1})'$ .So $l_{r_i}(Q_i'(R_i^{-1})'),i=1,2,$ $...,n$. $V_1V_2^{-1}=(\bigcap_{i=1}^{n}l_{r_i}(Q_i'))(\bigcap_{i=1}^{n}l_{r_i}(R_i'))^{-1}=(\bigcap_{i=1}^{n}l_{r_i}(Q_i'))(\bigcap_{i=1}^{n}l_{r_i}(R_i'))^{-1})=(\bigcap_{i=1}^{n}l_{r_i}(Q_i'))(\bigcap_{i=1}^{n}l_{r_i}((R_i^{-1})')$

(Lmark f the paper[4]) $\subset\bigcap_{i=1}^{n}l_{r_i}(Q_i')l_{r_i}((R_i^{-1})')=\bigcap_{i=1}^{n}l_{r_i}(Q_i'(R_i^{-1})')$ (Remark 1.3(3) of the paper[3]) $\subset$

$(\bigcap_{i=1}^{n}l_{r_i}(Q_i'))(\bigcap_{i=1}^{n}l_{r_i}(R_i^{-1})')$ From definition 22 in the paper[8] $(G,l_L(\delta)$ is a topological group.

End of proof.

Theory 3.6 Let (L^{G_1},δ_1) and (L^{G_2},δ_2) be LF topological groups.If $f_\bullet:(L^{G_1},\delta_1)\to(L^{G_2},\delta_2)$ is a LF homomorphism, general mapping $f:(G_1,l_L(\delta_1)\to(G_2,l_L(\delta_2)$ which induces $f_\bullet$ is a general homomorphism.

Proof: It is enough to prove that f is continuous.

Let $\varphi(\delta_2)=\{l_r(B):B\in\delta_2,r\in p(L)\}$ be subbasis of $(G_2,l_L(\delta_2))$ .It is easy to prove that $f^{-1}(l_r(B))=l_r(f_\bullet^{-1}(B))$ for $\forall l_r(B)\in\varphi(\delta_2)$ .Because of continuity of $f_\bullet$ we can get that $f_\bullet^{-1}(B)\in\delta_1$.

Therefore $l_r(f_\bullet^{-1}(B))\in l_L(\delta_1)$ In other words $f^{-1}(l_r(B))\in l_L(\delta_1)$ .This shows that f is generally continuous.

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